

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Notes on Charge-Neutral
Self-Consistent Plasmas and Fields

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ABSTRACT

In Part I the Vlasov equations are reduced to a general form for solving initial value problems. These equations are then studied from the point of view of type and of important scale lengths.

It is noted that transverse flows depending on only one space variable depend on c/ω_{pe} for small mass ratio but c/ω_{pi} is the natural length for all other cases. Steady flows on a scale length c/ω_{pi} are studied as asymptotic limits. Estimates show that the fields and currents cannot grow faster than $c/\omega_{pe} A$ where A is Alfvén speed unless the electrons are hot.

In Part II the time-dependent one-dimensional equations with transverse field are investigated by various methods. The asymptotic treatment of electrons is determined. Correctly posed problems for the initial-boundary value problems are discussed and also some limiting cases. This study is made in Lagrangian coordinates. The results of computations for weak Riemann problems where there is no breaking and rarefaction piston problem are presented along with the methods involved and justifications for certain assumptions such as neglecting third moments.

Finally, a model for strong shocks is presented in Part III. It is shown that under rather weak assumptions the final state is uniquely determined by the initial stage.

Introduction

These notes represent a variety of results all based on a charge neutral collisionless self-consistent model of a plasma of two species, electrons and ions, in an electromagnetic field. These requirements mean that we are dealing with two distribution functions each satisfying Boltzmann's equation without a collision term. The forces are the electromagnetic forces ruled by Maxwell's equations assuming the speed of light to be infinite. Thus the net charge is zero and there is no displacement current.

In Part I, §1 the equations are reformulated to lead to an algorithm for finding a solution as a function of time. The limiting case of zero density is briefly considered in §3. In §4 the interesting case in which the mass ratio of the two particles is regarded as asymptotically small is studied with a view to determining the important length scales. Vital differences depending on the geometry of the fields are brought out. In §5 and §6 asymptotic properties are pursued. The most interesting result for the time dependent case, §6, is that in the zero pressure case we must have density proportional to field magnitude on each field line.

In Part II the one-dimensional problem with transverse field is considered in some detail. Actually to compute this problem one must use asymptotic theories or some other hypothesis to make it tractable. Various possibilities for electrons are considered. For ions the simplest notion is to discard the higher than second moments. Some computations were carried out and in

part checked for their error, see §2 for the general results and §6 for the detailed comment on the figures.

In §4 the rather delicate question of what is a correctly posed problem to correspond with some physical problems is looked into.

§5 and §7 discuss the computational method for the differential equations and the third moments respectively.

Part III hazards a possible approach to the structure of strong shocks. This may very well involve turbulence which is itself so complicated that we cannot study strong shocks by using it. However some qualitative properties would give us useful shapes in principle and this part deals with a shock model using a minimum of special properties. It is a model in two parts. The first, §2, is a quasi-structure for an inner shock. The second, §3, is for the exterior overall shock structure for a situation where there is an inner shock.

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1. The General Equations of Motion

We consider a general two-component charge neutral gas of ions and electrons with distribution functions $f_i(\vec{u}, \vec{x}, t)$, charges ε_i and masses m_i , $i = 1, 2$, respectively. The self-consistent equations of motion are then,

$$f_{it} + \vec{u} \cdot \nabla_{\vec{x}} f_i + m_i^{-1} \varepsilon_i (\vec{E} + \mu \vec{u} \times \vec{B}) \cdot \nabla_{\vec{u}} f_i = 0, \quad i = 1, 2,$$

$$\text{curl } \vec{B} = \mu \vec{J}$$

$$\text{curl } \vec{E} = -\vec{B}_t$$

$$\tilde{q} = 0, \quad \text{div } \vec{B} = 0$$

where $\vec{E}$, $\vec{B}$ are the electric and magnetic fields, the current is $\vec{J} = \sum \int \varepsilon_i f_i \vec{u} |du|$ and the charge $\tilde{q} = \sum \tilde{q}_i = \int \sum \varepsilon_i f_i |du|$.

Introducing the length $\sqrt{m_1 m_2 / |\varepsilon_1 \varepsilon_2| \rho^* \mu}$ where ρ^* is some ion density and normalizing the other variables, we obtain with $\varepsilon_1 = -\varepsilon_2$

$$(1) \quad f_{it} + u \cdot \nabla_{\vec{x}} f_i + \begin{Bmatrix} \varepsilon \\ -\varepsilon^{-1} \end{Bmatrix} (E + u \times B) \cdot \nabla_u f_i = 0 \quad \begin{cases} \text{with } i = 1 \text{ use } \varepsilon \\ \text{with } i = 2 \text{ use } -\varepsilon^{-1} \end{cases}$$

where $\varepsilon = \sqrt{m_1/m_2} = \sqrt{m_-/m_+}$ and

$$\text{curl } B = J$$

$$(2) \quad -\text{curl } E = B_t$$

$$q_1 = q_2, \quad \text{div } B = 0$$

with

$$(3) \quad J = \varepsilon \int (f_1 - f_2) u |du| .$$

$$(4) \quad q_i = \int f_i |du| .$$

From the equations it is clear which quantities are vectors.

With these equations we shall have certain data given at $t = 0$ which we do not yet specify. We shall also keep in mind the case in which data is given on a surface $\mathcal{B}$ and we are concerned with solutions in its exterior or interior.

Our first object is to reduce these equations to the form,

$$(5) \quad U_t = A(U, V)$$

$$DV = G(U)$$

where A is a non-linear partial differential operator in the space variables and an integral operator in the velocity variables; D is a linear differential operator on V with coefficients depending on U and containing no time derivatives. D also is invertible provided the density of the gas does not tend to zero anywhere.

The advantage of this form is that we can also write it as

$$(5^*) \quad U(t) = U(0) + \int_0^t A(U, V) dt$$

$$V = D^{-1}G(U)$$

where $U(0)$ are specified.

From this form one is led naturally to various difference schemes which may or may not be stable.

This method was developed by Morton [3] in his one-dimensional, two-fluid study of compression waves and collisionless shocks. He found conditions for a stable difference scheme provided the solution existed and he also considered cases where only weak solutions with shocks occur.

We note that our equations are not in the desired form (5) because E occurs only in one of the equations, $B_t = \text{curl } E$, and not in any equations without a time derivative.

The desired equations are found by taking moments of (1) with respect to the velocity variables and rearranging the variables. We proceed to do this:

Integrating (1) with respect to u we get the two mass conservation equations:

$$q_{it} + \text{div} \int u f_i |du| = 0$$

or introducing the notation

$$(6) \quad \langle Q \rangle_i = \int Q(u) f_i du ,$$

$$(7) \quad \frac{\partial}{\partial t} q_i + \text{div} \langle u \rangle_i = 0 .$$

Multiplying (1) by u and integrating we obtain the six momentum equations:

$$(8) \quad \frac{\partial}{\partial t} \langle u \rangle_i + \text{div} \langle uu \rangle_i + \left\{ \begin{matrix} \epsilon \\ -\epsilon \end{matrix} - 1 \right\} (-Eq_i - \langle u \rangle_i \times B) = 0 .$$

Here uu denotes the tensor $u_k u_\ell$, $k, \ell = 1, 2, 3$.

For future use we also obtain here the next highest moment equations by multiplying by uu . Thus

$$(9) \quad \frac{\partial}{\partial t} \langle uu \rangle_i + \text{div} \langle uuu \rangle_i + \left\{ \begin{matrix} \varepsilon \\ -\varepsilon \end{matrix} -1 \right\} (-E \cdot \langle \nabla_u uu \rangle_i - \langle (u \times B) \cdot \nabla_u uu \rangle_i) = 0.$$

Note $\text{div} \langle uuu \rangle = \Sigma \frac{\partial}{\partial x_j} \langle u_k u_\ell u_j \rangle$, $E \cdot \langle \nabla_u uu \rangle_i = E_k \langle u_\ell \rangle_i + E_\ell \langle u_k \rangle_i$

and finally $\langle (u \times B) \cdot \nabla_u uu \rangle_i = \langle (u \times B)_k u_\ell + (u \times B)_\ell u_k \rangle_i$.

We introduce the density ρ and the mean fluid velocity U by

$$(10) \quad \rho = \int (f_1 + \varepsilon^2 f_2) |du| = \langle 1 \rangle_1 + \langle \varepsilon^2 \rangle_2 = (1 + \varepsilon^2) \langle 1 \rangle_1$$

$$\rho U = \langle u \rangle_1 + \varepsilon^2 \langle u \rangle_2$$

where (2) and (4) were used.

From (2) and (3) we then have $\text{curl } B = \langle \varepsilon u \rangle_1 - \langle \varepsilon u \rangle_2$ or, introducing the mean ion and electron velocities

$$(11) \quad \begin{aligned} \langle u \rangle_1 &= \langle 1 \rangle_1 U_1 = (1 + \varepsilon^2)^{-1} (\rho U + \varepsilon \text{curl } B) \\ \text{or } U_1 &= U + \varepsilon \frac{\text{curl } B}{\rho}, \\ \langle u \rangle_2 &= \langle 1 \rangle_2 U_2 = (1 + \varepsilon^2)^{-1} (\rho U - \varepsilon^{-1} \text{curl } B) \\ \text{or } U_2 &= U - \varepsilon^{-1} \frac{\text{curl } B}{\rho}. \end{aligned}$$

From (7), (10), (4) we see that there is really only one equation for conservation of mass:

$$(12) \quad \rho_t + \operatorname{div} (\rho \mathbf{U}) = 0 .$$

From (8) and (2) we obtain by linear combinations the equations for the total momentum $\langle \mathbf{u} \rangle_1 + \varepsilon^2 \langle \mathbf{u} \rangle_2 = \rho \mathbf{U}$ and the current $\langle \mathbf{u} \rangle_1 - \langle \mathbf{u} \rangle_2 = \varepsilon^{-1} \operatorname{curl} \mathbf{B}$:

$$(13) \quad (\rho \mathbf{U})_t + \operatorname{div} (\langle \mathbf{u} \mathbf{u} \rangle_1 + \varepsilon^2 \langle \mathbf{u} \mathbf{u} \rangle_2 - \operatorname{curl} \mathbf{B} \times \mathbf{B}) = 0$$

and

$$(14) \quad \frac{\partial}{\partial t} (\operatorname{curl} \mathbf{B}) + \varepsilon \operatorname{div} (\langle \mathbf{u} \mathbf{u} \rangle_1 - \langle \mathbf{u} \mathbf{u} \rangle_2 - \rho (\mathbf{E} + \mathbf{U} \times \mathbf{B})) \\ + \frac{1 - \varepsilon^2}{\varepsilon} \operatorname{curl} \mathbf{B} \times \mathbf{B} = 0$$

where we have used (11).

After some calculation we find from (13) and (11)

$$(15) \quad (\rho \mathbf{U})_t + \operatorname{div} (\rho \mathbf{U} \mathbf{U} + \mathbf{M}_1 + \varepsilon^2 \mathbf{M}_2) + \operatorname{curl} \mathbf{B} \cdot \nabla \frac{\operatorname{curl} \mathbf{B}}{\rho} - \operatorname{curl} \mathbf{B} \times \mathbf{B} = 0$$

where $\mathbf{M}_i = \langle (\mathbf{u} - \mathbf{U}_i)(\mathbf{u} - \mathbf{U}_i) \rangle_i$.

From (14) and the equation for $\mathbf{B}_t$ in (2) we obtain

$$\mathbf{B}_t = -\operatorname{curl} \mathbf{E} = \operatorname{curl} (\mathbf{U} \times \mathbf{B}) - \operatorname{curl} \frac{1}{\rho} (\operatorname{curl} \mathbf{B})_t \\ - \varepsilon \operatorname{curl} \rho^{-1} \operatorname{div} (\langle \mathbf{u} \mathbf{u} \rangle_1 - \langle \mathbf{u} \mathbf{u} \rangle_2) \\ - \frac{1 - \varepsilon^2}{\varepsilon} \operatorname{curl} \frac{1}{\rho} (\operatorname{curl} \mathbf{B} \times \mathbf{B}) .$$

Next introducing the variables $\mathbf{F}$, $\mathbf{U}^*$ by

$$(16) \quad \begin{aligned} \rho U^* &= \text{curl } B \\ \rho F &= B + \text{curl } U^* \end{aligned}$$

into the preceding equation we obtain

$$(17)^* \quad (\rho F)_t = \text{curl} [U \times B + \rho^{-1} (\text{div } \rho U) U^* - \varepsilon \rho^{-1} \text{div} (\langle uu \rangle_1 - \langle uu \rangle_2) - \frac{1-\varepsilon^2}{\varepsilon} U^* \times B] .$$

Returning again to equation (14) and using (16) we can find an equation for E using from (2) the expression for B_t . This yields

$$(18)^* \quad E + \rho^{-1} \text{curl curl } E = -U \times B + \varepsilon \rho^{-1} \text{div} (\langle uu \rangle_1 - \langle uu \rangle_2) + \frac{1-\varepsilon^2}{\varepsilon} U^* \times B .$$

We can put (17)* and (18)* in better form by using along with $\rho_t + \text{div } \rho U = 0$,

$$\begin{aligned} \text{div} (\langle uu \rangle_1 - \langle uu \rangle_2) &= \text{div} (M_1 - M_2) - \frac{1-\varepsilon^2}{\varepsilon^2} \rho U^* \cdot \nabla U^* \\ &+ \frac{1}{\varepsilon} U \cdot \nabla (\rho U^*) + \frac{1}{\varepsilon} \rho U^* \cdot \nabla U + \frac{\rho}{\varepsilon} (\text{div } U) U^* . \end{aligned}$$

Thus

$$\begin{aligned} (17) \quad \rho (F_t + U \cdot \nabla F) &= B \cdot \nabla U - \text{curl} (U \cdot \nabla) U^* + (U \cdot \nabla) \text{curl } U^* + \text{div } U \text{curl } U^* \\ &- \text{curl} (U^* \cdot \nabla U) + \frac{1-\varepsilon^2}{\varepsilon} \text{curl} ((U^* \cdot \nabla) U^* \\ &- (U^* \times B)) - \text{curl} \frac{\varepsilon}{\rho} \text{div} (M_1 - M_2) \end{aligned}$$

and

$$(18) \quad E + \rho^{-1} \text{curl curl } E = -U \times B + \frac{\varepsilon}{\rho} \text{div } (M_1 - M_2) - \frac{1-\varepsilon^2}{\varepsilon} (U^* \cdot \nabla U^* - U^* \times B) + U \cdot \nabla U^* + U^* \cdot \nabla U + \rho^{-1} (\text{div } \rho U) U^* .$$

From equation (17)* we only note further that

$$(19) \quad \frac{\partial}{\partial t} \text{div } \rho F = 0 .$$

We also put (15) in the usual form, using (12),

$$(20) \quad \frac{\partial U}{\partial t} + U \cdot \nabla U + \rho^{-1} \text{div } (M_1 + \varepsilon^2 M_2) + U^* \cdot \nabla U^* - U^* \times B = 0 .$$

We note a few interesting facts about our set of equations which are now, (1) for f_i and (12), (16), (17), (18) and (20).

I. (1), (12), (17) and (20) can all be written in conservation form. For steady one-dimensional flows $mF_T + \frac{\varepsilon}{\rho} \text{div } (M_1 - M_2)$ is constant as well as the mass flux m and momentum flux. Here subscript T indicates transverse component with appropriate sign.

II. It is not a priori clear that if there is a solution of (1), (12), (16), (17) and (20) that it is a solution of (1) and (2). This can be proved under rather general conditions for one-dimensional flow.

III. The highest derivatives of U^* (essentially the electron speed) that occur are second derivatives in (17) and these occur only in the $\text{curl } (U^* \cdot \nabla U^*)$ term.

The equations (1), (12), (16), (17), (18), (20) are now in the desired form (5) as far as the time derivatives are concerned. The variables 'U' are f_i , ρ , ρU , F and the variables 'V' are B , U^* , E .

Initial data: From the general form it is reasonable to prescribe in our case f_i , ρ , ρU , and F initially with ρ , f_i compatible. However F is a somewhat artificial variable and we can prescribe instead B provided $\text{div } B = 0$. Then from (16)

$$\rho F = B + \text{curl } \frac{1}{\rho} \text{curl } B$$

initially and hence $\text{div } F = 0$ at $t = 0$. Thus from (19)

$$(21) \quad \text{div } \rho F = 0 .$$

Physically we would also expect to require $\text{curl } B = J$ initially. We shall return to this point in discussing the computations.

The next question is whether we can solve for B , U^* and E .

Solving for B , U^* and E : We regard F and ρ as given in (16).

By (21) we may write

$$(22) \quad \rho F = \text{curl } A$$

and A may be regarded as known. Let Ψ be a vector and ϕ a scalar that together satisfy

$$(23) \quad \begin{aligned} -\Delta \Psi &= \rho(A - \Psi) + \rho \nabla \phi \\ \rho \Delta \phi + \nabla \rho \cdot \nabla \phi &= -\text{div } \rho A + \nabla \rho \cdot \Psi \end{aligned}$$

and which have square integrable gradients in infinite space.

We shall show that the solution exists for $\rho \neq 0$ later. Meanwhile we set $B = \text{curl } \Psi$, $U^* = A - \Psi + \nabla\phi$, and find that (16) is satisfied as follows: From (23),

$$\Delta \text{div } \Psi = \rho \text{div } \Psi$$

and hence since $\text{div } \Psi \rightarrow 0$ at ∞ , $\text{div } \Psi \equiv 0$ by the maximum principle. Therefore $\text{curl } B = \text{curl } \text{curl } \Psi = -\Delta\Psi = A\rho - \rho\Psi + \rho\nabla\phi = \rho U^*$ and

$$\text{curl } U^* = \text{curl } (A - \Psi) = \rho F - B$$

as required. Furthermore $\text{div } B = 0$ also. Hence the problem of solving for B and U^* is reduced to solving (23).

If $(\rho, F) \rightarrow (\rho_0, F_0)$ fast enough as $x \rightarrow \infty$ in one direction, say the $-x$ direction (or upstream), we may take

$$A = xA_1 + yA_2 + zA_3 + A^*$$

A_1, A_2, A_3 are all constant vectors and $A^* \rightarrow 0$ rapidly at ∞ . We set

$$\Psi = -(xA_1 + yA_2 + zA_3) + \Psi^*.$$

Then we find that our equations (23) become

$$(24) \quad \Delta\Psi^* - \rho\Psi^* = \rho A^* - \rho\nabla\phi$$

$$(25) \quad \rho\Delta\phi + \nabla\rho \cdot \nabla\phi = -\text{div } \rho A^* - \text{div } (\rho xA_1 + \rho yA_2 + \rho zA_3) \\ + \nabla\rho \cdot (-xA_1 - yA_2 - zA_3) + \nabla\rho \cdot \Psi^*$$

or

$$(26) \quad \rho \Delta \phi + \nabla \rho \cdot \nabla \phi = g + \nabla \rho \cdot \Psi^*$$

where $g \rightarrow 0$ as fast as $\rho - \rho_0$, $\nabla \rho$ etc.

These equations, (24) and (26), can be solved by iteration if ρ is almost constant, $\rho \sim \rho_0 \neq 0$ or existence can be established by a complete continuity argument along with uniqueness as follows:

Let us first prove uniqueness if ∇B lies in L^2 and F is given. To do this we return to equation (16) with $F = 0$ and find that since $B \cdot \text{curl } U^* = \text{div } Q - \sum_i \left| \frac{\partial B}{\partial x_i} \right|^2$ if we integrate over space we find $\int \sum_i \frac{1}{\rho} \left| \frac{\partial B}{\partial x_i} \right|^2 dx + \int B \cdot B dx = 0$. The only solution of the homogeneous equation is $B = 0$.

Here we note for future reference that $Q = -\text{div } B \times U^*$ and hence for the uniqueness of boundary value problems we should require

$$(27) \quad \text{The normal component } B \times U^* \text{ prescribed on surface } \mathcal{B}.$$

We consider next integral equations corresponding to (24) and (26) restricting ourselves to $\rho \neq 0$. The fundamental solution for the operator $\Delta - \rho_0$ is $r^{-1} \exp -\sqrt{\rho_0} r$ and for Δ is r^{-1} with $r = |x - x^1|$. Thus

$$(28) \quad \phi = \int \frac{1}{r} \left(-\frac{\nabla \rho \cdot \nabla \phi}{\rho} + \frac{\nabla \rho}{\rho} \cdot \Psi^* + g \right) |dx| = L_1(\phi, \Psi^*) + f,$$

$$\Psi^* = \int r^{-1} \exp(-\sqrt{\rho_0} r) ((\rho - \rho_0) \Psi^* - \rho \nabla \phi + \rho A^*) |dx| = L_2(\phi, \Psi^*) + h.$$

We note that $\nabla(\frac{1}{r})$ and $r^{-1} \exp -\sqrt{\rho_0} r$ both lie in L^2 at ∞ . Thus one finds with $\|\phi\|^2 = \int |\nabla\phi|^2 |dx|$ and $\|\Psi\|^2 = \int (\Psi^* \cdot \Psi^*) |dx|$ that by classical methods

$$\|\nabla L_1(\phi, \Psi^*)\|^2 \leq \alpha(\|\phi\|^2 + \|\Psi^*\|^2)$$

$$\|L_2(\phi, \Psi^*)\|^2 \leq \beta(\|\phi\|^2 + \alpha\|\Psi^*\|^2)$$

where α is an absolutely constant multiple of the maximum of $\nabla\rho/\rho$ and $\rho - \rho_0$ and β is bounded. Hence L_1 and L_2 combined are contraction operators for $\nabla\phi, \Psi$ if α is sufficiently small, that is the pair

$$\nabla L_1(\phi, L_2(\phi, \Psi)) \quad , \quad L_2(L_1(\phi, \Psi), \Psi)$$

are contraction operators. Thus the standard iteration scheme will converge.

Existence also follows because the operators are completely continuous. This is easily proved by standard methods. We have already established uniqueness so now we have existence provided $\nabla\rho/\rho$ is bounded.

This argument also works for some boundary value problems.

The electric field E can be found by inverting the equation for $\text{curl curl } E$ and noting that $\text{div } \rho E$ is given as a function of the other variables.

2. Reduction of the Momentum Transfer Equations

We note further $\text{div} \langle uuu \rangle = \frac{\partial}{\partial x_j} \langle u_j u_k u_l \rangle$
 $= \frac{\partial}{\partial x_j} (M_{jkl} + M_{jk} U_l + U_k M_{jl} + U_j M_{kl} + \langle U_j U_k U_l \rangle)$ where the subscript i
has been suppressed and

$$M_{jkl}^i = \langle (u_j - U_{ij})(u_k - U_{ik})(u_l - U_{il}) \rangle_i$$

$$M_{kl}^i = M_l = \langle (u_k - U_{ik})(u_l - U_{il}) \rangle_i$$

where U_{ij} are the three components of U_i . We then expand (9) and recall that along with the other equations we must obtain an identity if all the second and third moments vanish. Thus we obtain $B = (B_1, B_2, B_3)$, and with ϵ_{kir} vanishing if $k = i$, $i = r$ or $k = r$, and equalling ± 1 as kir is an even or odd permutation of 1, 2, 3,

$$\begin{aligned} & \frac{\partial}{\partial t} M_{kl} + \frac{\partial}{\partial x_j} (M_{jkl} + M_{jl} U_{ik} + M_{jk} U_{il} + M_{kl} U_{ij}) \\ & - U_{il} \frac{\partial}{\partial x_j} M_{kj} - U_{ik} \frac{\partial}{\partial x_j} M_{lj} - \delta (\epsilon_{kir} M_{il} B_r + \epsilon_{lir} M_{ik} B_r) = 0 \end{aligned}$$

where M_{kl} , U_k etc. is for ions or electrons according as δ is ϵ or $-\frac{1}{\epsilon}$.

3. The Limiting Case as $\rho \rightarrow 0$

We observe that if $\rho \rightarrow 0$ at a finite point then our equations (24-26) are singular since $\nabla \log \rho = \nabla \rho / \rho$ becomes infinite. One needs either an existence theorem or to establish that $\rho \rightarrow 0$ only at ∞ . In the sheath problem, the simple Rosenbluth model [1] has

$\rho \sim \sqrt{x - x_c}$ and this coefficient is singular but in Sestero's models ρ vanishes only at ∞ , [2].

The other equations are all non-singular as $\rho \rightarrow 0$ and in principle give no difficulties, except that the Alfvén speed tends to ∞ . We note only that U^* must be bounded so $\text{curl } B \rightarrow 0$ but that F may be singular with ρF bounded. We see this by considering the equation for ρF instead of F .

4. The Limiting Case of ε Small and the Associated Wave Lengths

Here we observe that there are two distinct cases to consider:

We set

$$\text{curl } Q = \text{curl } (U^* \times \rho F - \frac{\varepsilon}{\rho} \text{div } M_2)$$

and the two cases are

$$\text{Case A: } \quad \text{curl } Q \equiv 0$$

$$\text{Case B: } \quad \text{curl } Q \neq 0 .$$

This crucial expression is found in the differential equation for F . It is the only term that becomes singular as $\varepsilon \rightarrow 0$. As shown in (17), this term is

$$\begin{aligned} - \text{curl } Q &= \text{curl } ((U^* \cdot \nabla) U^* - U^* \times B) - \text{curl } \frac{\varepsilon}{\rho} \text{div } M_2 \\ (29) \quad &= \text{curl } (U^* \times (B - \text{curl } U^*) - \frac{\varepsilon}{\rho} \text{div } M_2) \\ &= \text{curl } (U^* \times \rho F - \frac{\varepsilon}{\rho} \text{div } M_2) . \end{aligned}$$

Here we have used the fact that M_2 is $O(\frac{1}{\varepsilon^2})$ if the pressures of ions and the electrons are commensurable.

We first make an estimate for how large the currents and magnetic field can become:

From (17*) using (16)

$$B \cdot (\rho F)_t = - \operatorname{curl} \frac{Q}{\varepsilon} + O(1)$$

or by (16) dotted with B ,

$$\left(\frac{1}{2} |B|^2\right)_t - B \cdot \operatorname{curl} U_t^* = - \frac{1}{\varepsilon} B \cdot \operatorname{curl} Q + O(1) .$$

On integrating over space, we see that we may integrate by parts if we assume that all gradients vanish at large distances. Thus

$$(30) \quad \frac{\partial}{\partial t} \int \left[\frac{1}{2} |B|^2 + |\operatorname{curl} B|^2 \right] |dx| - U^* \cdot \operatorname{div} M_2 |dx| = O(1)$$

since

$$\begin{aligned} \int B \cdot \operatorname{curl} Q |dx| &= - \int \operatorname{curl} B \cdot (U^* \times F - \frac{\varepsilon}{\rho} \operatorname{div} M_2) |dx| \\ &= - \int \rho U^* \cdot (U^* \times \rho F - \frac{\varepsilon}{\rho} \operatorname{div} M_2) |dx| \\ &= \varepsilon \int U^* \cdot \operatorname{div} M_2 |dx| \end{aligned}$$

and

$$\int B \cdot \operatorname{curl} U_t^* |dx| = - \int \rho U^* \cdot U_t^* |dx| = - \frac{\partial}{\partial t} \int \rho |U^*|^2 |dx| + O(1) .$$

The term involving the electron pressure tensor represents of course the work done on the electrons.

Equation (30) asserts that provided $U^* \cdot \text{div } M_2 = o(\frac{1}{\varepsilon})$, i.e. the work done by the stresses in the direction of the current is $O(1)$ the fields remain finite.

On the other hand we see that $E \sim O(\frac{1}{\varepsilon})$ unless Q itself vanishes. The quantity ρF is a modified field. If the gradients of B are small $\rho F \sim B$. Hence if Q vanishes we have a particular configuration somewhat like a force-free field.

Thus it is to be expected that there will be two kinds of flows — those in which the term in $\text{curl } Q$ dominates and those where it is small.

The natural wave lengths: The two natural wave lengths are for:

Case (A): The basic unit used here for normalization, i.e., c/ω_{pe} . If we let $\varepsilon \rightarrow 0$ and drop ε compared to 1 we have

$$\rho_t + \text{div } \rho U = 0$$

$$U_t + U \cdot \nabla U + \rho^{-1} \text{div } (M_1 + \varepsilon^2 M_2) + U^* \cdot \nabla U^* - U^* \times B = 0$$

$$\rho(F_t + U \cdot \nabla F) = + B \cdot \nabla U - \text{curl } (U \cdot \nabla) U^* + U \cdot \nabla \text{curl } U^*$$

$$(31a) \quad + (\text{div } U) \text{curl } U^* - \text{curl } U^* \cdot \nabla U + \text{curl } \frac{\varepsilon}{\rho} \text{div } M_2$$

$$E + \rho^{-1} \text{curl curl } E = -U \times B - \frac{\varepsilon}{\rho} \text{div } M_2 + \frac{1}{\varepsilon} U \cdot \nabla U^*$$

$$+ \frac{1}{\varepsilon} U^* \cdot \nabla U + \rho^{-1} \text{div } (\rho U) U^*$$

$$\rho F = B - \text{curl } U^* , \quad \rho U^* = \text{curl } B .$$

A special case which has good asymptotic properties occurs if $\text{curl} (\mathbf{U}^* \cdot \nabla \mathbf{U}^*) \neq 0$ but $\text{curl} (\mathbf{U}^* \times \mathbf{B} + \varepsilon^2 \text{div } \mathbf{M}_2) = 0$. We introduce as a new length scale $\varepsilon^{1/3} x$ and in this unit obtain

$$\rho_t + \text{div } \rho \mathbf{U} = 0$$

$$\mathbf{U}_t + \mathbf{U} \cdot \nabla \mathbf{U} + \rho^{-1} \text{div} (\mathbf{M}_1 + \varepsilon^2 \mathbf{M}_2) - \frac{\text{curl } \mathbf{B}}{\rho} \times \mathbf{B} = 0$$

$$\rho (\mathbf{F}_t + \mathbf{U} \cdot \nabla \mathbf{F}) = + \mathbf{B} \cdot \nabla \mathbf{U} + \text{curl} \left(\frac{\text{curl } \mathbf{B}}{\rho} \cdot \nabla \frac{\text{curl } \mathbf{B}}{\rho} \right) - \text{curl} \frac{\varepsilon}{\rho} \text{div} (\mathbf{M}_1 - \mathbf{M}_2)$$

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} = \frac{\varepsilon^{4/3}}{\rho} \text{div} (\mathbf{M}_1 - \mathbf{M}_2) - \frac{\text{curl } \mathbf{B}}{\rho} \cdot \nabla \frac{\text{curl } \mathbf{B}}{\rho} - \varepsilon^{-2/3} \left(\frac{\text{curl } \mathbf{B}}{\rho} \times \mathbf{B} \right)$$

and

$$\rho \mathbf{F} = \mathbf{B} .$$

The advantage of this model is that the electrons are truly adiabatic and the ions see mainly electric field.

Case (B): The basic length (and time-scale) should balance the two sides of (17). Hence t and x should be scaled up by a factor ε , i.e., $x = \varepsilon x^1$, to the ion length, c/ω_{pi} . The equations reduce, on dropping the primes again, and neglecting terms of order ε to

$$\rho_t + \text{div } \rho U = 0$$

$$U_t + U \cdot \nabla U + \rho^{-1} \text{div } (M_1 + \varepsilon^2 M_2) - \frac{\text{curl } B}{\rho} \times B = 0$$

$$(31b) \quad \rho U^* = \varepsilon \text{curl } B, \quad \rho F = B$$

$$E + U \times B = \frac{-\varepsilon^2}{\rho} \text{div } M_2 - \frac{\text{curl } B}{\rho} \times B$$

$$B_t + \text{div } UB = +B \cdot \nabla U + \text{curl } \left(\frac{\varepsilon^2}{\rho} \text{div } M_2 - \frac{\text{curl } B}{\rho} \times B \right).$$

Note that the electric field is of order $1/\varepsilon$ in Case (A) unless $Q = 0$ but is of order 1 in Case (B).

5. Motion of Particles

Case (A): The ions are governed by the electric field. The electrons may be treated adiabatically only for thermal speeds less than order $1/\varepsilon$. This means that the corresponding β -factor for the electrons must be small. There is a special adiabatic invariant. Strictly speaking $Q \sim U^* \times \rho F$.

This case occurs for example in studying the structure of low β transverse shocks, where there is only one space variable. See Morton [3], Auer, Hurwitz, and Kilb [4], Kilb [5], Morawetz [6], Marder [7]. See also Gardner, Goertzel, Grad, etc. [8], Morikawa and Gardner [9]. In fact, this is the appropriate length scale for any field perpendicular to the axis of a single independent (space) variable.^{*} In general for two space variables even in a transverse field ω_{pi} is the scale length, Case (B).

^{*} This length first appeared in the Adlam-Allen wave [10].

Case (B): Here $E = O(1)$ in ε and

$$E \cdot B = \frac{\varepsilon^2}{\rho} B \cdot \text{div} (M_1 - M_2)$$

which as we shall see may be $O(1)$. However we may treat the electrons adiabatically even if their pressure is finite for the following reasons (despite the usual requirement $E \cdot B = O(\varepsilon)$):

Suppose the pressure is finite since the mass is small. Most electrons must have large velocities, i.e., to have $\varepsilon^2 \int f u^2 |du|$ finite and $\int f |du|$ also of order 1 we must have $u = U/\varepsilon$ where U is finite.

For the length scale of Case B, c/ω_{pi} , the equation (1) for the electrons becomes

$$\varepsilon^2 f_t + \varepsilon U \cdot \nabla f - (\varepsilon E + U \times B) \cdot \nabla_U f = 0 .$$

This has the same asymptotic behavior as

$$U \cdot \nabla f - \frac{1}{\varepsilon} (\varepsilon E + U \times B) \cdot \nabla_U f = 0 ;$$

the equation describing particles in a magnetic field B , electric field εE and a mass to charge ratio ε . Since $\varepsilon E \cdot B = O(\varepsilon)$ we can use the standard adiabatic theory [15] and thus $\varepsilon^2 M_2$ is

$$\begin{pmatrix} p & 0 & 0 \\ 0 & p_{\perp} & 0 \\ 0 & 0 & p_{\perp} \end{pmatrix}$$

in the framework of the field line as x-axis with $p_{\perp} = p_0/|B|^2$.

p_0 is constant on the field lines. To order ε the energy is conserved for each particle and thus $p = 3p_0(1 - \frac{2}{3|B|^2})$. Note that since $\varepsilon^2 M_1 = O(\varepsilon^2)$ we have

$$\begin{aligned} E \cdot B &\sim \frac{1}{\rho} B \cdot \text{div } \varepsilon^2 M_2 \\ &\sim \frac{1}{\rho} B \cdot \frac{p_0}{R} \frac{\partial}{\partial s} R \left(1 - \frac{2}{3|B|^2} \right) \end{aligned}$$

where R is the radius of curvature of the field line and s is arc length on it.

This situation occurs for shocks in oblique fields, see Morton [3], Drummond [11], and also Kever and Morikawa [12]. It also occurs for a transverse shock with finite β . The one-dimensional solution to this problem requires a discontinuity to admit a shock. R. Morse [13] has examined the ion stability numerically in a related model and found evidence of cellular behavior ahead of a shock.

One should also mention the original wave of this type, the Saffman wave [14].

6. Steady-state Solutions near Limit $\varepsilon \rightarrow 0$

Case (B). Ion Length

We investigate some asymptotic properties of the limiting case in the steady state. The underlying mathematical question is: What are the asymptotic connections between our original equations and the limiting case (B) described by (31). We pose this only for an equilibrium state. We are looking for the

properties of solutions whose natural length scale is the ion length c/ω_{pi} .

Preserving the correction terms in the many-dimensional case we have for an equilibrium solution from (20) with $U = 0$:

$$\rho^{-1} \operatorname{div} (M_1 + \varepsilon^2 M_2) + U^* \cdot \nabla U^* + U^* \times B = 0$$

$$\rho U^* = \operatorname{curl} B, \quad \operatorname{div} B = 0.$$

Since $U^* \cdot \nabla U^*$ is of higher order we see that the characteristics are no longer the field lines as they are if this term is missing. Instead the characteristics satisfy $U^* \cdot dx = 0$ or $\operatorname{curl} B \cdot dx = 0$ and hence are in the direction of the current.

We examine the case of zero pressure

$$U^* \cdot \nabla U^* + U^* \times B = 0$$

$$\operatorname{div} B = 0.$$

Scaling up to the ion length c/ω_{pi} introduces the factor ε and we have

$$U^* \cdot \nabla U^* + \frac{1}{\varepsilon^2} (U^* \times B) = 0$$

$$\rho U^* = \operatorname{curl} B, \quad \operatorname{div} B = 0$$

where U^* has also acquired the ε -factor.

If we regard U^* as a velocity the first equation is for the motion of a particle in a steady magnetic field and its energy U^{*2} is a constant U_0^{*2} along each current line. There is an

adiabatic invariant if ϵ is small which is $U^{*2} - (U^* \cdot B/|B|)^2/|B|$. U^* consists of a component parallel to the magnetic field which is varying with the field derivatives and a component perpendicular in a direction that moves with period $O(\frac{1}{\epsilon})$ around the magnetic field line. That is, if the second derivatives of B are bounded as $\epsilon \rightarrow 0$.

$$(32) \quad U^* = (U_0^{*2} - |B| \mu)^{1/2} \frac{B}{|B|} + (|B| \mu)^{1/2} j + O(\epsilon)$$

where $j = \sin \frac{\xi}{\epsilon} k + \cos \frac{\xi}{\epsilon} l$; k, l being a pair of unit orthogonal vectors perpendicular to B , $\xi = \int |B| ds$ along the field line; U_0^*, μ are constants on each field line.

The Limiting Solutions as $\epsilon \rightarrow 0$ with Finite Second Derivatives

Assume there is a limit with B having finite second derivatives as $\epsilon \rightarrow 0$. The adiabatic form, (32), may be regarded as a differential equation for B of the form

$$\text{curl } B = \rho g(B) .$$

If there is a solution to this equation then, since g contains the rapidly oscillating vector j , some of the second derivatives of B will tend to infinity as $\epsilon \rightarrow 0$ unless the amplitude factor of the unit vector j tends to zero. Thus in (32), $\mu \sim 0$ for ϵ small. Hence $U^* \sim U_0^* B/|B|$ with U_0^* constant on each field line.

On the other hand $\text{div } \rho U^* = 0$ or $B \cdot \nabla \rho/|B| = 0$. Hence $\rho/|B|$ is constant on each field line in the limit as the Larmor radius $\rightarrow 0$ if the pressure is zero.

Note that if we proceed to the asymptotic limit directly we obtain no condition on ρ only

$$U^* \times B = 0$$

or

$$U^* \parallel B$$

subject to $\text{div } B = 0$. Hence there are many solutions of this limit which are not true limits.

It seems unlikely that this solution will be stable since it is embedded in a family of solutions (32) (with μ not necessarily zero) which are highly oscillatory.

Let us consider next the case of finite pressure. Introducing a fictitious electric field $\tilde{E}$ we see that the new equilibrium equations are

$$\tilde{E} = - \frac{1}{\rho} \text{div } (M_1 + \varepsilon^2 M_2)$$

$$U^* \cdot \nabla U^* = \frac{1}{\varepsilon^2} (\tilde{E} + U^* \times B)$$

$$\text{div } B = 0 .$$

In order that $U^* = O(1)$ one can show that $\tilde{E} \cdot B = O(\varepsilon)$. Hence the pressure tensor is restricted to satisfy

$$B \cdot \text{div } (M_1 + \varepsilon^2 M_2) = O(\varepsilon) .$$

Then with the guiding center interpretation we find that in order to have a solution of $O(1)$ in ε ,

$$U^* = - \frac{\tilde{\mathbf{E}} \times \mathbf{B}}{|\mathbf{B}|^2} + \lambda \frac{\mathbf{B}}{|\mathbf{B}|} \quad \text{where} \quad \lambda^2 = U_o^{*2} - \left| \frac{\tilde{\mathbf{E}} \times \mathbf{B}}{|\mathbf{B}|^2} \right|^2$$

where U_o^* is constant along the current lines.

The condition $\text{div curl } \mathbf{B} = \text{div } \rho \mathbf{U}^*$ then leads to

$$\mathbf{B} \cdot \nabla \left(\frac{\lambda \rho}{|\mathbf{B}|} \right) - \text{div} \frac{\rho \tilde{\mathbf{E}} \times \mathbf{B}}{|\mathbf{B}|^2} = 0$$

or

$$\mathbf{B} \cdot \nabla \left(\frac{\lambda \rho}{|\mathbf{B}|} \right) - \text{curl} (\rho \tilde{\mathbf{E}}) \cdot \frac{\mathbf{B}}{|\mathbf{B}|^2} + \text{curl} \left(\frac{\mathbf{B}}{|\mathbf{B}|^2} \right) \cdot \frac{\rho \tilde{\mathbf{E}}}{|\mathbf{B}|^2} = 0 .$$

Thus we have a differential equation for the pressure tensor $\rho \tilde{\mathbf{E}} = \mathbf{M}_1 + \varepsilon^2 \mathbf{M}_2$. Or we can regard this equation as a restriction on ρ corresponding to the zero pressure case.

1. Equations

We consider in greater detail and for numerical study the case in which the only independent variables are: x and t and the field B is in the z -direction. The dependence of f_i on the third velocity component drops out.

The set of differential equations reduces with $\bar{U} = (\bar{u}, \bar{v})$, by noting that $\text{curl } B$ has a component only in the y -direction, to

$$\rho_t + (\rho \bar{u})_x = 0$$

$$\bar{u}_t + \bar{u} \bar{u}_x + \rho^{-1} \frac{\partial}{\partial x} (M_1 + \varepsilon^2 M_2)_x \text{ comp} + \rho^{-1} B_z B_{zx} = 0$$

$$\bar{v}_t + \bar{u} \bar{v}_x + \rho^{-1} \frac{\partial}{\partial x} (M_1 + \varepsilon^2 M_2)_y \text{ comp} = 0$$

$$(I) \quad F_{zt} + \bar{u} F_{zx} = -\varepsilon \rho^{-1} \frac{\partial}{\partial x} \frac{1}{\rho} \frac{\partial}{\partial x} (M_1 - M_2)_y \text{ comp}$$

$$-U_{2x}^* = \left(\frac{1}{\rho} B_{zx} \right)_x = B_z - \rho F_z, \quad U_1^* = U_z^* = F_2 = F_1 = 0, \quad B_2 = B_1 = 0$$

$$E + U \times B - \frac{1}{\rho} E_{2xx} = \varepsilon \rho^{-1} (M_1 - M_2)_x + \bar{u} U_x^* + \rho^{-1} (\rho \bar{u})_x U^* + \frac{1-\varepsilon^2}{\varepsilon} (U^* \times B)$$

and the original equations (1) for f_i reduce to

$$(II) \quad f_{it} + u f_{ix} + (E + u \times B) \cdot \nabla_u f = 0, \quad i = 1, 2.$$

We note that $U_1 = U_2$ in the x -direction.

Thus the momentum equations (32) for $M_{11}^i = N_{\pm}$, $M_{12}^i = M_{\pm}$, $M_{22}^i = L_{\pm}$ reduce to

$$N_{\pm t} + N_{\pm x} \bar{u} + 3N_{\pm} \bar{u}_x - \delta_{\pm} 2BM_{\pm} + N_{3\pm x} = 0$$

$$(III) \quad M_{\pm t} + M_{\pm x} \bar{u} + 2M_{\pm} \bar{u}_x + N_{\pm} \bar{v}_x - \delta_{\pm} (BL_{\pm} - \rho FN_{\pm}) + N_{21\pm x} = 0$$

$$L_{\pm t} + (L_{\pm} \bar{u})_x + 2M_{\pm} \bar{v}_x + \delta_{\pm} 2\rho FM_{\pm} + N_{12\pm x} = 0$$

where the third moments are

$$N_{3\pm} = \int f_{\pm} (u - \bar{u})^3 \, du dv$$

$$N_{21\pm} = \int f_{\pm} (u - \bar{u})^2 (v - \bar{v}_{\pm}) \, du dv$$

$$N_{12\pm} = \int f_{\pm} (u - \bar{u}) (v - \bar{v}_{\pm})^2 \, du dv$$

and

$$\delta_+ = \varepsilon, \quad \delta_- = -\varepsilon^{-1}.$$

Finally we will want to know the electric fields which are found by (18) to be

$$(IV) \quad \begin{aligned} E_1 &= -B\bar{v} + \varepsilon \rho^{-1} (N_+ - N_-)_x - \frac{1-\varepsilon^2}{2\varepsilon} \rho^{-1} (B^2)_x \\ E_{2x} &= -B_t. \end{aligned}$$

Theoretically, we could now set up a numerical scheme for solving these equations in the following way. We go forward in time by using a difference scheme on (I) and (III) to obtain new values of ρ , $\bar{u}$, $\bar{v}$, F , $L_{\pm}$, $M_{\pm}$, $N_{\pm}$ using the old values of ρ , $\bar{u}$, $\bar{v}$, F , $L_{\pm}$, $M_{\pm}$, $N_{\pm}$, B , and the third moments differenced in x to obtain space derivatives. The new values of B are then obtained by solving a second order ordinary differential equation for F (the

last equation of (I)). New values of the third moments are obtained by following the characteristics of the two transport equations for f , i.e. following the particles and estimating the moments. To do this one needs the electric fields which are obtained from (IV). However, this scheme is unrealistic, principally for two reasons. Firstly, when ε has its physically significant value approximately $1/60$, one sees from (III) that we would need time steps that are small compared to $1/\varepsilon$ but we know that significant waves will take place on the given time scale of order 1.

The simplest way to resolve this dilemma is to set the third moments of the electrons equal to zero. This can be completely justified if the adiabatic approximation is valid, that is, if the electron pressure is small, i.e., if the thermal velocity is $O(1)$ or even $\ll 1/\varepsilon$. In this case, see [6], the distribution is symmetric in $u - \bar{u}$, $v - \bar{v}$, to lowest order in ε and not only may N_{3-} , N_{21-} and N_{12-} be neglected but also M_- is small.

From (II) for the electrons we see that for ε small we must have to lower order

$$(1) \quad BL - F\rho N = 0 .$$

Using (1) and eliminating M_- from the first and third electron moment equations we obtain a differential equation for L_- (or N_-). We find after using the conservation of mass, the first equation of (I),

$$(2) \quad \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{B\rho^{-5}L^2}{F} = 0$$

or $BV^5L^2F^{-1}$ is constant on a fluid path $\frac{dx}{dt} = \bar{u}$. Using (2) in the first electron moment equation we obtain

$$(3) \quad \epsilon M_- = \frac{\epsilon^2}{2F} \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) (\rho^{-1} L_-) .$$

Note that we need ϵM_- to find F in (1c).

In what follows we shall assume that in any event we can neglect the third moments of the electrons, see Appendix A. We shall either use, from (1), (2), (3), assuming ϵ is small

$$(4) \quad (B\rho^{-5}F^{-1})^{1/2} L_- = N_0 , \quad (FB^{-1}\rho^{-3})^{1/2} N_- = N_0$$

where N_0 is constant on $dx/dt = \bar{u}$, and

$$(5) \quad \epsilon M_- = \frac{1}{2} F^{*-1} N_0 \left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) (FB^{-1}\rho^3)^{1/2}$$

or we shall use the differential equations:

$$(6) \quad \begin{aligned} N_{-t} + N_{-x} \bar{u} + 3N_- \bar{u}_x + 2\epsilon^{-1} B M_- &= 0 \\ M_{-t} + M_{-x} \bar{u} + 2M_- \bar{u}_x + N_- \bar{v}_x + \epsilon^{-1} (B L_- - \rho F N_-) &= 0 \\ L_{-t} + (L_- \bar{u})_x + 2M_- \bar{v}_x - 2\epsilon^{-1} \rho F M_- &= 0 . \end{aligned}$$

The second problem is connected with finding the third moments for the ions. It is reasonable to consider only problems in which the third moments vanish initially. A power series expansion in time shows that the third moments develop very slowly — in the limit $\epsilon \rightarrow 0$ like t^4 . On the other hand, it is very difficult to compute their derivatives accurately by following

particles. We have therefore set the third moments equal to zero in solving (I) and (III) but as a check computed their values from the resulting fields.

The equations have been solved with the third moments of the ions set equal to zero. See Chew, Goldberger and Low [16] for a similar model.

In all cases where the differential equations with the third moments set equal to zero appear to have a continuous solution the third moments are so small that they cannot have any qualitative effect. In cases where the solution is about to break (as the zero-pressure solitary wave breaks if the particle velocity goes to zero) it appears that the third moments acquire an oscillation but the magnitudes of the moments will not alter the situation and stop the breaking.

2. Summary of Numerical Observations

From (I) we see as expected that the 'charge separation' electric field E_1 is $O(1/\epsilon)$. Hence the motion of the ions is dominated by this field. If, in (II), we set the third moments of the ions equal to zero we find since M_+ and $(BL_+ - F\rho N_+)$ are small

$$(N_+\rho^{-3})_t + \bar{u}(N_+\rho^{-3})_x = 0$$

where we have used the conservation of mass $\rho_t + (\rho\bar{u})_x = 0$ from (Ic).

In fact, this checks with our numerical calculations remarkably well. A similar argument which yields a different pressure tensor L_+ , M_+ , N_+ can be carried out for an axial pinch, see Marder [7]. His results also involve different moments for the electrons.

It is clear that dropping the third moments in (II) should be a good approximation if the thermal speed of the ions is small compared to the other velocities in the problem. What is remarkable is that even with thermal speeds of the order of .5 which corresponds to $\beta = .5$ a rarefaction wave which cuts B in half seems to satisfy these conditions very well, see Figures 37, 42, 43. Then the largest third moment at time $T = 4$ is still only .02. Smaller values of β are still better.

On the other hand, if we generate a compression wave that does not break in the case where we set the third moments equal to zero we find that the possible strength of the maximum compression declines very rapidly with pressure. For example, if the compression is induced by a five percent change in magnetic field the thermal speed must be less than $1/4$ of the Alfvén speed. If there is no initial magnetic field gradient but a pressure gradient (no density gradient) which induces compression, the wave breaks with thermal speeds of about .2 or $\beta = .08$. For example, the neglected third moments in case initially $\beta = .14$ are approximately 10^{-4} at their maximum, i.e. an order of magnitude smaller than the u_{th}^4 where u_{th} is thermal speed.

Let us recollect here that when a wave is 'saved' from breaking by viscosity two properties help: 1) the amount of

viscosity and, ii) the fact that this coefficient multiplies a higher derivative and hence the restoring effect is amplified as the wave tends to steepen. Here the third moments occur with the same order of derivative as the second and the magnitude does not grow adequately. Furthermore any apparent oscillation (it is not clear that this represents some real fluid behavior) will have an alternating effect on the breaking.

3. Equations in Lagrangian Coordinates with Neglected Third Moments

We introduce the Lagrangian variables T, ξ by $\frac{\partial}{\partial T} = \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}$, $\frac{\partial}{\partial \xi} = \frac{1}{\rho} \frac{\partial}{\partial x}$. The differential equations simplify, with $V = 1/\rho$, to

$$\begin{aligned}
 \frac{\partial V}{\partial T} &= \frac{\partial \bar{u}}{\partial \xi} \\
 \frac{\partial \bar{u}}{\partial T} &= - \frac{\partial}{\partial \xi} \left(\frac{B^2}{2} + N_+ + \varepsilon^2 N_- \right) \\
 (7) \quad \frac{\partial F}{\partial T} &= - \frac{\partial^2}{\partial \xi^2} (M_+ - M_-) \\
 \frac{\partial \bar{v}}{\partial T} &= - \frac{\partial}{\partial \xi} (M_+ + \varepsilon^2 M_-) \\
 \frac{\partial^2 B}{\partial \xi^2} &= BV - F .
 \end{aligned}$$

The moment equations are with $\delta_+ = 1$, $\delta_- = -\varepsilon^{-1}$,

$$\begin{aligned}
(8,9) \quad \frac{\partial}{\partial T} V^3 N_{\pm} &= 2\delta_{\pm} B V^3 M_{\pm} \\
\frac{\partial}{\partial T} V^2 M_{\pm} &= \delta_{\pm} (B V^2 L_{\pm} - F V N_{\pm}) - \frac{V N_{\pm}}{1+\varepsilon} \frac{\partial \bar{V}}{\partial \xi} \\
\frac{\partial}{\partial T} V L_{\pm} &= -2\delta_{\pm} F M_{\pm} - \frac{2M_{\pm}}{1+\varepsilon} \frac{\partial \bar{V}}{\partial \xi} .
\end{aligned}$$

For the electrons alternatively for low temperatures, (4,5),

$$\begin{aligned}
(10) \quad L_{-} &= g^{1/2}(\xi) F^{1/2} B^{-1} V^{-5/2} , \quad N_{-} = \frac{V B L_{-}}{F} = g^{1/2}(\xi) V^{-3/2} B^{1/2} F^{-1/2} \\
\varepsilon M_{-} &= \frac{\varepsilon^2}{2F} (V L_{-})_T .
\end{aligned}$$

The electric fields satisfy

$$(11) \quad E_1 = -B\bar{V} + \frac{\partial}{\partial \xi} (\varepsilon(N_{+} - N_{-}) - (\frac{1-\varepsilon^2}{2\varepsilon})B^2) , \quad \frac{\partial E_2}{\partial \xi} = -V \frac{\partial B}{\partial T} + \bar{u} \frac{\partial B}{\partial \xi} .$$

4. Correctly Posed Initial Boundary Problems and Numerical Method

Simply by counting we observe that for the full set of equations (I)-(III) we may prescribe $f_{\pm}$ initially along with B . On a boundary $x = h(t)$ we may prescribe, if the region of interest corresponds to $x - h(t) > 0$, the value of B and the values of $f_{\pm}$ for $u > \frac{dh}{dt}$. If the region corresponds to $x \rightarrow \infty$ we require B to be bounded at ∞ and we may prescribe $f_{\pm}$ for $u < 0$, $x = \infty$.

In making approximations by discarding the third moments and using the differential equations for all second ones we can prescribe initially the six moments, V , $\bar{u}$, $\bar{v}$, and B and on a boundary

$\xi = \text{constant}$ we can prescribe B (or one other quantity). The same is true if we use the asymptotic system for the electrons. We have reasoned on the basis of counting alone but the characteristic speeds for the full system of the eleven equations indicate that more conditions must be imposed.

To obtain the characteristic speeds from this cumbersome system we rewrite the equation for F as

$$F_T = G_\xi$$

and

$$\alpha G_T = G - (M_+ - M_-)\xi$$

and consider the limiting case as $\alpha \rightarrow 0$. This way we may miss speeds^{*} but on the other hand we do not appear to introduce any extraneous ones. Since roots are multiple anyway we cannot use standard characteristic theory. But we can get some feel of what is required. The characteristic speeds are then $0, \infty, \pm\sqrt{N_+ + \varepsilon^2 N_-}$.

This suggests that we ought to prescribe some conditions on the boundary and some recent work indicates that the appropriate quantities are M_+ , B and $\bar{u}$. The last one falls out if we look at the fluid dynamic limit, $B = 0$ and the condition on B is seen from its equation, see also Marder [7].

* An example of this is given by the system $u_t = p_{xx}$, $p_t = u$. Rewriting this as a limiting case of $u_t = q_x$, $\alpha q_t = p_x - q$, $p_t = u$ with $\alpha \rightarrow 0$ we find the three speeds $0, 0, 0$ from the formal matrix. But written as $u_t = q_x$, $q_t = u_x$, $p_t = u$ we obtain $\pm 1, 0$, as speeds.

We consider next correctly posed problems for the low temperature case, i.e. where we use (4,5) for the electron moments.

If we use low temperature equations then the equation for F may be written as

$$\frac{\partial F}{\partial T} = - \frac{\partial^2}{\partial \xi^2} M_+ + \left(\frac{\varepsilon^2}{2F} \frac{\partial}{\partial T} (V L_-) \right)_{\xi\xi} .$$

To solve this equation along with the others we might consider the last term as small, i.e. $\varepsilon^2 L_-$ is small and continue using our singular perturbation as we did in deriving the expressions (4,5). Thus we set $F = F_0 + F_1$, $B = B_0 + B_1$

$$(12) \quad \frac{\partial F_0}{\partial T} = - \frac{\partial^2 M_+}{\partial \xi^2} , \quad \frac{\partial F_1}{\partial T} = \left(\frac{\varepsilon^2}{2F_0} \left(\frac{g_{F_0}}{B_0 V^3} \right)_T^{1/2} \right)_{\xi\xi}$$

$$\frac{\partial^2 B_0}{\partial \xi^2} = B_0 V - F_0 , \quad \frac{\partial^2 B_1}{\partial \xi^2} = B_1 V - F_1 .$$

These equations replace the equations for F and B in (I).

Combining them again we get

$$(13) \quad \frac{\partial^2 F}{\partial T} = - \frac{\partial^2 M_+}{\partial \xi^2} + \left(\frac{\varepsilon^2}{2F_0} \left(\frac{g_{F_0}}{B_0 V^3} \right)_T^{1/2} \right)_{\xi\xi}$$

$$\frac{\partial F_0}{\partial T} = - \frac{\partial^2 M_+}{\partial \xi^2}$$

$$\frac{\partial^2 B}{\partial \xi^2} = B V - F$$

$$\frac{\partial^2 B_0}{\partial \xi^2} = B_0 V - F_0 .$$

The computations for low pressure electrons were carried out using a variation of these equations which will be discussed later. See also Marder [7] for a better scheme.

Another possibility is to examine the system directly. We set

$$P = \frac{\varepsilon_{VL}^2}{2F} \left(\frac{F_T}{2F} - \frac{B_T}{2B} - \frac{3V_T}{2V} \right)$$

$$= \frac{V_L \varepsilon^2}{2F} \frac{\partial}{\partial T} \log (\varepsilon_{VL}^2)$$

then by (7)

$$F_T = P_{\xi\xi} - \varepsilon M_{+\xi\xi} \quad \text{and} \quad P_{\xi\xi} - \varepsilon M_{+\xi\xi} = F \left(\frac{B_T}{B} + \frac{3V_T}{V} + \frac{4FP}{\varepsilon_{VL}^2} \right) .$$

Next set

$$(14) \quad B_T = R$$

so that from (7) for B differentiated, (7) for F and (7) for V we have

$$(15) \quad R_{\xi\xi} = RV + B\bar{U}_{\xi} + P_{\xi\xi} - \varepsilon M_{+\xi\xi}$$

and also

$$(16) \quad P_{\xi\xi} = F \left(\frac{R}{B} + \frac{3\bar{U}_{\xi}}{V} + \frac{4FP}{\varepsilon_{VL}^2} \right) + \varepsilon M_{+\xi\xi} .$$

We now compute the characteristic speeds as limits for these equations with the variables R, P in addition to V, U, E, F, $\bar{V}$, B, N_+ , M_+ , L_+ .

The possible speeds are $\pm \sqrt{(N_+ + \varepsilon^2 N_-)}$, 0 (six times) and ∞ (twice). Thus two data should be prescribed on $\xi = \text{constant}$. One of these could be $\bar{u}$. For the other we might choose M_- on $\xi = \text{constant}$; say $M_- = 0$ which implies that a certain symmetry prevails at the edge. This would be resolved in a boundary layer of scale length ℓ determined by the coefficient of P in (16), $\ell = \varepsilon^2 (V_{L_-})^{1/2} / 2F$, i.e. by the thermal speed of the electrons with the factor ε . This is essentially the thermal speed of the ions if the two pressures are both roughly the same. Ahead of the boundary layer one may reasonably expect the asymptotic behavior described by (13). Recall that it is consistent with low pressure electrons to neglect the third moments and the boundary layer both at $\xi = \text{constant}$ and at $t = 0$.

Breaking Waves

The computations show that if we assume our adiabatic formulas for the electrons or neglect the third moments only weak compression waves do not break. (Rarefaction waves yield small third moments and are quite consistent.)

5. The Computational Method*

A grid of points (i, j) $i = 0 \dots N$, $j = 0 \dots M$ is introduced in ξ, T space and the partial differential equations (7), (8) and (9) if used are solved in the following way. Every T derivative

* All the computations described in this and subsequent sections were programmed and carried out by Antoinette Wolfe. The machine was the CDC 6600.

is replaced by the difference D which satisfies

$2kD U(i,j) = U(i,j+1) - U(i,j-1)$ and the first order ξ derivative by d where

$$2hd U(i,j) = U(i+1,j) - U(i-1,j) .$$

Except in the equation for B the second order derivative is replaced by d^* with

$$4h^2 d^* U(i,j) = U(i+1,j) + U(i-1,j) - 2U(i,j) .$$

At the edge $\xi = \text{constant}$ the values of $M_+ - M_-$ needed in (7) are extended by interpolation. Thus we can proceed one time step forward in a regular way, i.e. all variables except B evaluated at $j+1$, $i = 0 \dots h$ can be computed from their values at (i,j) , $(i,j-1)$ for all i . To solve for B we use the method of Richtmyer [3]. The only difference is that $F = 1$ in his problem. The method makes use of the fact that the solutions may be written as $C(B_h - 1) + D$ where $B_h - 1$ satisfies the corresponding homogeneous equation and tends to 1 at infinity and D satisfies the inhomogeneous equation. One prescribes B at the left hand mesh point to determine C.

To solve the adiabatic equations (10) instead of (9) we use a modification of (13). At each time step we solve for F assuming $M_- = 0$. We then use this new value of F to compute M_- from (10) replace in the equation (7) for F and obtain a corrected value of F. This is probably unstable if the pressure is too large.

This part of the computing scheme is accurate to second order. It was used on two types of problems.

a) Initial value problems in which the field $B(\xi, 0)$ (or pressure $N_+ + \varepsilon^2 N_-$) is prescribed at $T = 0$ and all other data kept constant where compatible* while $B(0, T)$ is kept fixed.

b) Boundary value problems (magnetic piston) in which all data are constant initially and $B(0, T)$ is prescribed.

The principal object of a) was to see how an initial 'discontinuity' would propagate. If our equations were those of magnetohydrodynamics the initial discontinuity would be resolved by a Riemann shock tube configuration with a shock going from the higher field or pressure into the lower field or pressure region and a rarefaction the other way.

In our problems this change is prescribed in the form

$$B(0, \xi) = B_0 + \alpha \tanh c(\xi - \xi_0) .$$

The parameter c fixes the steepness and because of the mesh sizes involved we cannot take c too small. The changes take place over approximately half the Larmor radius of the ion. For zero pressure the field first flattens and then generates waves which travel ahead as is expected from the magnetic piston problem, see Morton [3]. If the amplitude of the change in the initial B exceeds about 1.4 the wave breaks almost immediately.

This pattern is repeated if pressure is present but the wave breaks for far less increase in field.

* One sees from (11) and (7) we can either prescribe N_- or L_- constant initially but not both if $B \neq$ constant initially.

In problem b) a change in strength for B initially constant is prescribed at the end point. For compression waves the end effects were very strong in the third moments. Here we describe only rarefaction results.

6. Explanation of Corresponding Figures

Figures 1-5 show how the wave develops in the B field. The dotted line is the initial field. The solid line shows B at times $T = 0, 1, 2, 3, 4$. Beyond $T = 4$ there are reflection effects from the ends. Figure 6 shows how the ion temperature, initially .07, develops and Figure 7 a comparison with electron pressure (adiabatic formulas) at $T = 4$. The ions are hotter in the compression and cooler in the rarefaction. All later figures are at $T = 4$ to show maximum effect. Figure 8 shows $\rho^{-1} - 1$, the increase in specific volume. Figure 9 shows mean velocity. Note the sharp dip in this case. If U goes to zero or approximately sound speed the wave breaks. Figure 10 shows the mean velocity in the y-direction. Figure 11 the off-diagonal pressure tensor term is very small. Figure 12 shows how little the pressure in the y-direction changes in comparison with that in the x-direction. Figure 13 shows the special function F. Figure 14 confirms that the electric field in the x-direction is $O(1/\epsilon)$ in comparison with the y-field in Figure 15. Note there is a considerable end effect in the $E_2 = E_y$ field. Figures 16 and 17 show the third moments (see the next section for the method). The largest is barely of the order $(N_+)^{3/2}$ and shows no regular pattern. The ends for a length $1/2$

have been smoothed out. The condition of reflection used at the ends leads to very bad behavior and it is not known what suitable relaxation condition should be made.

Figures 18-25 show the same problem with half the rise in the field but $N_+ \sim .1$. Qualitatively the same phenomena occur. However the third moments are considerably smaller as seen in Figures 26 and 27, certainly negligible for any self-consistent requirement. Figure 28 shows the percentage error (maximum 2-4) in the density computed in the same way as the third moments and compared with the solution of the partial difference scheme. Figure 29 shows that in one place the particles collected and in another were missing — there were 80 in each mesh box to start with. The two points correspond with the two minima of the E field, see Figure 24.

Figures 30-35 show a similar Riemann problem with an initial step in pressure in a constant magnetic field. The same kind of phenomena occurs. The third moments however develop much more shape. Hopefully this problem could be solved self-consistently by improving the mesh size by a factor of say 5.

The maximum rarefaction was produced by cutting B to half its value. Figures 36-45 illustrate the results in this case. Note that the pressures, Figure 37, may be taken so that $\beta \sim 1$. Note from Figure 40 that the gas density has been halved also. The third moments are of the order of N_+^2 and negligible. One notices that N_{12+} has relatively little noise and its shape may be valid. Figures 44 and 45 show a remarkable uniformity of noise in the

density and the peaks in the displaced particles do not seem to be correlated to the fields. The model thus appears to be very effective for rarefaction waves.

The stability of any of these computations has not been investigated. For the purpose of computing the third moments such small time steps were taken ($k/h = .1$) that no apparent difficulties arose in this connection and certainly h/k was large compared to all characteristic speeds.

The time consuming computation is involved in computing the neglected third moments. The method is discussed in the next section.

7. Computations of the Third Moments

To compute the third moments of a distribution in a given field we follow the particles. In the case of the ions this means to solve the transport equation (Ia) by finding its characteristics (the particle paths) which satisfy, with $E = \epsilon E_x$,

$$\frac{dx}{dt} = u$$

$$\frac{du}{dt} = E + \epsilon v B$$

$$\frac{dv}{dt} = \epsilon E_y - \epsilon u B .$$

This was done by the Runge-Kutta method using a three point scheme.

Since E , E_y and B are only known at the mesh points of the Lagrangian system we transformed from x , t to ξ , T and interpolated the fields on the original mesh.

Note that this method also would work if we were using a completely self-consistent system with the third moments present. Each characteristic is computed a step forward after the field at the later time step has been found.

Let us assume now that the characteristics can be computed with an arbitrary degree of accuracy. Then (II) states that f_+ is constant on this characteristic.

At $T = 0$, the distribution f_+ is given at each mesh point and at each mesh point in some region of u,v -space.

To each mesh point, r, s in the initial velocity space we assign a weight factor $W_{rs} = f(u(r,s), v(r,s))$ and an area factor a_{rs} which together represent the element $fdudv$ corresponding to that mesh point. Then initially, if the density is 1,

$$(17) \quad \sum W_{rs} a_{rs} = 1 .$$

The initial pressure say in the x -direction is represented by

$$(18) \quad \sum W_{rs} u_{rs}^2 a_{rs} .$$

The mesh in the initial velocity space was taken as circular with, for example, 16 divisions of the angle and 4 or 5 radii. The largest radius difference was .1 but depended on the pressure. Increasing the number of points to 160 in the basic case increased the smoothness, i.e. the fluctuations were cut in half. This is an indication that in the future with better codes one might obtain smooth enough third moments for self-consistent computations by increasing the number of mesh points.

We have taken f either as a step function or as a Maxwellian in which case the density $\rho = 1$ normalized the Maxwellian and the exponent was determined from the pressure.

8. Estimating the Third Moments

A useful lemma for estimating is

Lemma: $\rho^{-1} f du dv d\xi$ is invariant along a characteristic in u, v, ξ, t -space.

To prove this one integrates the differential equation (II) over an arbitrary tube $\mathcal{T}$ generated by a set of characteristics. Thus, applying Green's theorem one has

$$\int_{\partial \mathcal{T}} f(t_n + u x_n + (E + \varepsilon v B) u_n + \varepsilon(E_y - u B) v_n) ds = 0$$

where t_n, x_n, u_n, v_n are the components of the normal on the boundary $\partial \mathcal{T}$ of $\mathcal{T}$. The factor vanishes on the characteristic boundary and choosing $\partial \mathcal{T}$ initially and at t to be $t = \text{const.}$ we see that $\int_{\sigma'} f du dv dx$ is independent of time. But $\int_{\sigma'} f du dv dx = \int_{\sigma} f \rho^{-1} du dv d\xi$. This proves the lemma.

Next we note that any moment

$$G = \int f(u - \bar{u})^{\ell} (v - \bar{v})^m du dv$$

may be written as

$$\begin{aligned}
G &= \frac{1}{H} \int_{-H}^{+H} g(u,v,\xi,t) f du dv d\xi + O(H^2) \\
&= \frac{1}{H} \int_{-H}^{+H} \rho g(f\xi^{-1}) du dv d\xi + O(H^2) .
\end{aligned}$$

We can evaluate this integral by writing it as $\sum_n \int_{\sigma_n} (\rho g) f \rho^{-1} du dv d\xi$. Here σ_n is a region of u,v,ξ -space containing a particle P_n . It is chosen* to be such that it is the end of the characteristic tube at t generated from $t = 0$ by the three-dimensional unit σ_n^* formed by the mesh unit in the initial velocity space and the interval $\xi^* - \frac{1}{2} h, \xi^* + \frac{1}{2} h$ where ξ^* is the initial position of P_n .

By our lemma,

$$\begin{aligned}
\int_{\sigma_n} (\rho g) \rho^{-1} f du dv d\xi &= \int_{\sigma_n^*} \rho g(\rho^{-1} f)^* du^* dv^* d\xi^* \\
&= \int_{\sigma_n^*} \rho g(\rho^{-1} f)^* dr^* d\theta^* d\xi^*
\end{aligned}$$

where $(\rho^{-1} f)^*$ is the value of $\rho^{-1} f$ at r^*, θ^*, ξ^* with r^*, θ^* polar coordinates in the initial velocity space. This integral may be approximated by $(\rho g)_n \rho^{-1}(\xi^*) w_{rs} h_{rs}$ with an error of K^3 . Here $K = \max(h, K_1)$ where K_1 is the maximum mesh width in r^*, θ^* -space, $(\rho g)_n$ is the value of ρg at the point P .

* Of course this choice cannot be made for particles too close to the boundary $\xi = H$. These particles will be denoted by Q_n and discussed later.

The particles Q_n would produce an error if dropped of the order h/H . In our computations where $h/H = 1/4$ we followed the following method which smooths out the end effect and reduces this error.

The method depended on the fact that

$$\int fg \, dudv = \int_{-H}^{+H} \frac{fg \, q(\xi') \, dudv \, d\xi'}{\int q(\xi') \, d\xi'} + O(H^2)$$

and q was chosen to vanish at $\xi' = \pm H$ and to be linear on $(-H, 0)$ and $(0, H)$. Then $\int fg \, dudv$ was approximated by

$$\sum_n (\rho g q)_n w_{rs} a_{rs}^{2H} \int_{-H}^{+H} q(\xi') \, d\xi'.$$

(In all our problems $\rho^{-1}(\xi^*) = 1$. Also H was taken to be $2h$.)

It was found as a check that the density computed this way never differed from that computed as V^{-1} in the p.d.e. problem by more than a few percent. When the density variation was as much as 20 percent the maximum error was 6 percent. The third moments were at their largest about 10^{-3} which was the same order as K^3 . It should be added that near the ends of the total ξ interval the error in the density could rise as high as 20 percent when the density had hardly changed.

The above estimates have assumed that the particle paths had negligible error. One should note that the error accumulates in time but in fact this will be of lower order than K^3 .

Appendix I - Finite Electron Pressure

For finite electron pressure the speed of most particles must be $O(1/\varepsilon)$ to guarantee that $N_- = \varepsilon^2 \int f(u-\bar{u})^2 du dv = O(1)$ and $\int f du dv = O(1)$. We make the change of scale $U = \varepsilon u$, $V = \varepsilon v$ and observe that the transport equation for the electrons becomes

$$f_{-t} + Uf_{-x} - (E + VB)f_{-u} + (\varepsilon E_y - UB)f_{-v} = 0$$

where we have set $E_x = E/\varepsilon$ in accordance with (Ie).

To lowest order in ε this equation reduces to $Uf_{-x} - (E + VB)f_{-u} + UBf_{-v} = 0$ and describes the transport in a sheath (i.e. $E_y = 0$), see Sestero [2]. The electrons have closed paths in (X, U, V) -space which satisfy

$$U \frac{dU}{dx} = -E - VB$$

$$\frac{dV}{dx} = B.$$

Setting $\int_{x^*}^x B dx = A$ we have then with $\phi = \int_{-\infty}^x E dx$

$$V = A + V^*, \quad \frac{U^2}{2} + \phi + \frac{A^2}{2} + V^*A = \frac{U^{*2}}{2} + \phi^*$$

where U^* , V^* are the values at $x = x^*$. The paths are closed^{*} since $A^2 \rightarrow +\infty$ for $x \rightarrow \pm\infty$ and ϕ is bounded. The particle motion

^{*} Note that the full equations in the steady state require every particle to go from $-\infty$ to $+\infty$, see [9].

given by the Hamiltonian has an ignorable coordinate to lowest order in ϵ . Applying the theory described by Gardner [8] we have as adiabatic invariant the area swept out by an orbit in the appropriate phase space. This area a is determined by its value at $x = x^*$. The distribution f_- is therefore to lowest order a function of a only. This problem, while theoretically possible, is difficult to compute.

Appendix II

Here we shall show that for a distribution of ions moving in a steady field we have the second moment $N = \int f(u-\bar{u})du dv$ behaving like ρ^3 , see Chew, Goldberger and Low [6]. This lends plausibility to the calculations with neglected third moments and small off-diagonal components of the stress tensor.

Let f satisfy

$$(1) \quad u f_x + E(x) f_u = 0 \quad \text{with} \quad f = e^{-k(u-\bar{u})^2} \quad \text{for} \quad x = \infty.$$

Then

$$\int f u^2 du dv \sim \int f u^2 du.$$

But the element udu is invariant in x . Hence $\int f u^2 du$ may be written as $\int f^* u(x) u^* du^*$ where f^* is, let us say, the value of f at ∞ for a particle with $u = u^*$ and $u(x)$ is its velocity at x . Then since $\frac{du}{dt} = u \frac{du}{dx} = E$ we have $u^2 = 2\phi + u^{*2}$ where $\phi = \int_{\infty}^x E dx$. Thus

$$\int f u^2 du = \int_{-\infty}^{+\infty} e^{-k(u^* - \bar{u})^2} (u^{*2} + 2\phi)^{\frac{1}{2}} u^* du^*$$

where we assume that E is such that there are no closed paths in (x,y)-space.

We set $k(u^* - \bar{u}) = \xi$, $N = \int f(u-U)^2 du$ where $U = \int f u du / \int f du$. By the conservation of mass $\int f u du = \text{const.}$ which we normalize to be $\bar{u}$ and $U = \bar{u}$ for $x = \infty$. Then

$$N + \bar{u}U = \int f u^2 du = \frac{1}{k} \int e^{-\xi^2} \left(\left(\bar{u} + \frac{\xi}{k} \right)^2 + 2\phi \right)^{\frac{1}{2}} \left(\frac{\xi}{k} + \bar{u} \right) d\xi.$$

We expand in inverse powers of k and find, since odd terms in ξ vanish,

$$\begin{aligned} N + \bar{u}U &= k^{-1} \bar{u} (\bar{u}^2 + 2\phi)^{\frac{1}{2}} \int e^{-\xi^2} d\xi \left(1 + \frac{3}{2} \frac{1}{\bar{u}^2 + 2\phi} \frac{\xi^2}{k^2} - \frac{1}{2} \frac{\bar{u}^2}{(\bar{u}^2 + 2\phi)^2} \frac{\xi^2}{k^2} + \dots \right) \\ &= \bar{u} (\bar{u}^2 + 2\phi)^{\frac{1}{2}} + \frac{1}{2} N_0 (\bar{u}^2 + 2\phi)^{-\frac{1}{2}} - \bar{u}^3 (\bar{u}^2 + 2\phi)^{-\frac{3}{2}}. \end{aligned}$$

But

$$\begin{aligned} U &= \bar{u} \left(\int f^* (u^{*2} + 2\phi)^{-\frac{1}{2}} u^* du^* \right)^{-1} \\ &= k \bar{u} \left(\int e^{-\xi^2} \left(\left(\bar{u} + \frac{\xi}{k} \right)^2 + 2\phi \right)^{-\frac{1}{2}} \left(\frac{\xi}{k} + \bar{u} \right) d\xi \right)^{-1} \\ &= (\bar{u}^2 + 2\phi)^{\frac{1}{2}} + \frac{3}{2} N_0 (\bar{u}^2 + 2\phi)^{-\frac{1}{2}} - \frac{3}{2} N_0 \bar{u}^3 (\bar{u}^2 + 2\phi)^{-\frac{3}{2}} \end{aligned}$$

similarly. Thus we find with $\rho U = \bar{u}$ to lowest order

$$N \sim N_o \bar{u}^3 (\bar{u}^2 + 2\phi)^{-\frac{3}{2}} + O(\frac{1}{k^4})$$

or

$$N \sim N_o \rho^3 + O(N_o^2)$$

as it was desired to show. Note that the neglected term is of order 4 in the thermal velocity.

1. The Shock Problem

Observations [J. Paul [17] etc. [18,19]] of transverse shocks indicate that there are two critical shock speeds or Mach numbers when the observed phenomena for transverse shocks change character. Let us call the three corresponding zones 1, 2, 3:

In zone 1, the flow is oscillatory with wave length as in case A,
 c/ω_{pe} .

In zone 2, the flow has a sharp initial jump, with oscillations
behind of wave length, c/ω_{pe} .

In zone 3, the flow has only a sharp internal jump, i.e. width $\ll$
 c/ω_{pe} . Aside from this jump $(\frac{|\nabla B|}{B})^{-1} = \text{length} = 0$ (c/ω_{pe}).

This description is fitted by Morton's model [3] in which the three zones satisfy the following criteria:

Zone 1. The flow is subsonic (using gas speed) and oscillatory.

Zone 2. The flow becomes supersonic in an internal shock but remains oscillatory.

Zone 3. The flow has no oscillations but a sharp internal viscous jump to supersonic speeds.

The statements describing zone 2 could be altered to fit the model of Morawetz [6]. That is, by introducing enough resistivity (nicknamed anomalous) the internal jump can be dispensed with and replaced by a strongly damped oscillation.

Morton and Morawetz are based on a gas law for the pressure tensors. Modifications of such models to treat the electrons

adiabatically and the ions by the electric field (see case A) have been made, see Marder [7] and Part II of this paper. The basic observations for zone 1 remain the same.

For a zone with a jump it seems reasonable to hypothesize an inner layer governed either by large charge separation fields or by turbulence or more probably by both.

2. Turbulent Inner Shock

In a purely turbulent model of an internal shock we note that even if the flow is highly dependent on time and space but is confined to a narrow region any equation that can be written in conservation form will yield a possible jump condition.

We examine the equation for U_t in (31), Part I. The term $\rho U^* \cdot \nabla U^*$ may be written as $\text{div } \rho U^* U^*$. Thus the conservation law is to be found by integrating with respect to time and space, combining with the conservation of mass and obtaining

$$\left[\int \rho U \, dx \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \int \text{div} [\rho U U + M_1 + \epsilon^2 M_2 + \rho U^* U^*]_- \, dx dt - \int_{t_1}^{t_2} \int (B \cdot \nabla B - \frac{\nabla |B|^2}{2}) \, dx dt = 0 .$$

The space integration is over the turbulent layer to some variable point. Since the layer is supposed thin the first term is small and the only contribution from integrating div terms will be the x-component. Ordinary one fluid theory differs from this only in the term $\rho U^* U^*$ and in the off diagonal terms in the pressure tensor.

Thus in an average sense with respect to time and y,z-space (indicated by bars), the average momentum is conserved:

$$\left[\overline{\rho U_x U} + \overline{(M_1 + \epsilon^2 M_2)_x} + \overline{\rho U_x^* U^*} - \overline{B_x B} + \frac{\overline{|B|^2}}{2} k \right] = 0$$

where k is a unit vector in the x-direction and subscript x is the x-component. The conservation of momentum in the x-direction then yields since $B_x = 0$

$$\left[\overline{\rho U_x^2} + \overline{(M_1 + \epsilon^2 M_2)_x} + \overline{\rho U_x^{*2}} + \overline{B^2/2} \right] = 0 .$$

To obtain an effect like viscosity there would have to be a term that changed sign with $\partial U_x / \partial x$. This leads us to the following ansatz

We hypothesize that turbulence is produced by an instability that depends on the magnitude of the electric field in the x-direction (i.e. two stream instability of one kind or another). Furthermore, we assume that the average effect is to keep B , $\text{curl } B$ constant. This modifies the average x-momentum equations to the form

$$\left[\overline{\rho U_x^2} + \overline{(M_1 + \epsilon^2 M_2)_x} + \overline{\rho^{-1}} (\text{curl } B)_x^2 + \overline{\nu E_x} \right] = 0$$

where $[\]$ means the jump from say x_0 to x , and $\overline{(M_1 + \epsilon^2 M_2)_x}$ is the scalar pressure. Here ν is some proportionality factor. Since the layer is assumed thin the force governing all motion is the electric field.

Let $d\phi/dx = E_x$. Then all flow quantities, $\bar{\rho}$, $\bar{U}_x$, $\bar{M}_{1x}$, $\bar{M}_{2x}$ can be seen to be functions of ϕ . The momentum equation is then, dropping bars,

$$v\phi_x = p(\phi)$$

which integrates to

$$x = v \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{p(\phi)}$$

where

$$p(\phi) = \rho U_x^2 + (M_1 + \varepsilon^2 M_2)_x + \rho^{-1}(\text{curl } B)_x^2 - M$$

with M the total momentum. It is also reasonable to assume $p(\phi) > 0$ in which case the shock layer has monotonically increasing ϕ in the mean.

Note that the shock width is proportional to v and roughly the jump in potential ϕ . The state ahead is given by a zero of

$$p(\phi) = \rho U_x^2 + (M_1 + \varepsilon^2 M_2)_x + \rho^{-1}(\text{curl } B)_x^2 - M$$

determined by one upstream state.

One hopefully looks for a second root to find the state behind the shock. For this a necessary condition is that $p'(\phi)$ vanishes somewhere. Using [6], to describe the ions in a potential field along with their reflections and somewhat crudely treating the electrons for this argument only as an ordinary gas, $p \propto \rho^\gamma$, one finds, since by [6],

$$\begin{aligned}\frac{\partial}{\partial \phi} (\rho U_x^2 + M_{1x}) &= \frac{\partial}{\partial \phi} \int f u^2 du dv = \frac{\partial}{\partial \phi} \int f u^* \sqrt{u^{*2} - 2\phi} du^* dv^* \\ &= - \int f u^* \sqrt{u^{*2} - 2\phi} du^* dv^* = -\rho ,\end{aligned}$$

that

$$p'(\phi) = -\rho + \rho_\phi (c_-^2 - \rho^{-2} (\text{curl } B)_x^2)$$

where c_-^2 is the gas sound speed of the electrons. Thus $p'(\phi)$ will vanish if c_-^2 is large enough and $\rho_\phi > 0$. At the maximum value of ϕ , again by [6], ρ_ϕ will vanish and then become positive. If it becomes large enough it can produce a second root to $p(\phi) = 0$. Thus one may reasonably anticipate this type of transition for strong enough shocks.

However, independently we may examine the states behind and in front of this special layer drawing from this very crude model only the minimum information. We proceed to do this.

3. Transverse Shock Structure for Larger Mach Numbers with an Inner Shock

On the basis of the model described in [6] and the layer described in §2, we hypothesize certain properties, of a steady shock, i.e. at rest in an appropriate frame. The flow, which depends on x only, consists of four regions:

$x < x_0$: All ions with u negative are being turned around by the magnetic field. The electric field is $O(1)$. The basic width is half the Larmor radius or about c/ω_{pi} .

$x_0 < x < x_1$: The ions are being thrown back by an increasing electric field $O(1/\epsilon)$. The density and mean x-velocity are functions of ϕ , and, see [6], the maximum value of ϕ for the whole range of x . Here $d\phi/dx = \epsilon E_x$. The basic length is c/ω_{pe} , our basic length. The electrons are essentially adiabatic. All we shall need to use is that the mean electron velocities are given by the guiding center.

$x_1 < x < x_2$: A black box in which we assume that ϕ increases monotonically and across which B and $\text{curl } B$ are constant. (It could be a turbulent boundary layer as in §2.)

$x_2 < x < \infty$: The ions are again governed by ϕ since $E = O(1/\epsilon)$. The density is a function of ϕ determined by the maximum of ϕ . Basic length c/ω_{pe} . The electrons are assumed to have mean velocities satisfying the appropriate guiding center equations. The electron pressure is assumed finite here so this is a bad approximation near x_2 . However for large x where the fields are varying slowly in the present length scale it is a valid model unless the distribution of the electrons is highly skewed.

Considerable information can be obtained on the basis of this model, in particular, the final values of pressure, field etc. can be determined theoretically. So at this point it is important to restate what is assumed: (1) The nature of the domain $x_1 < x < x_2$ as described and (2) that the mean velocities of the electrons in $x_2 < x < \infty$ satisfy the guiding center equations.

Let U, V be the (x, y) velocities of the guiding center motion. Then from II say, setting $E_x = E/\epsilon$

$$\varepsilon U \frac{dV}{dx} = +E_y - UB, \quad 0 = \frac{E}{\varepsilon} + VB$$

which yields $\varepsilon V = -E_x/B$. From $\text{curl } E = B_t = 0$ we have $E_y = \text{const.} = 1$ in our present normalization.

Thus

$$BU = 1 - U \, d(E/B)/dx$$

$$\varepsilon V = E/B.$$

Reverting, see (3) and (4), Part I, to $\text{curl } B = J$ instead of using (I), Part II, we find $\frac{dB}{dx} = -\varepsilon V \rho_- = -\varepsilon V \rho = -\varepsilon m \frac{V}{U} = -\frac{\varepsilon V}{U}$ since our mass flux $m = \rho U$ is constant and normalized to 1. Thus

$$(1) \quad U \frac{dB}{dx} = \frac{E}{B}, \quad U \frac{d}{dx} \left(\frac{E}{B} \right) = -1 + UB$$

or

$$(2) \quad U \frac{d}{dx} \left(U \frac{dB}{dx} \right) = -1 + UB.$$

It is easy to show by conservation of mass, charge and charge neutrality that U is also the mean x -velocity of the ions and that $U = \rho^{-1}$ is a function of ϕ . Since each ion with x -velocity u satisfies $\frac{du}{dt} = u \frac{du}{dx} = E = \frac{d\phi}{dx}$ to lowest order.

The actual function is different for the two regions under consideration, i.e. $x_0 < x < x_1$ and $x_2 < x < \infty$. We denote these two functions by $R_1(\phi)$ and $R_2(\phi)$ respectively. It is technically impossible to determine $R_1(\phi)$ explicitly. The governing conditions are described in Morawetz [6]. For our purposes it suffices to know that both functions are uniquely determined by giving the

maximum value of ϕ provided ϕ is monotonic up to x_2 . Integrating (1), from x_2 , B and ϕ are determined as functions of x or $B = B_2(\phi, \phi_{\max}, B(x_2))$ for $x > x_2$ or

$$(3) \quad \phi = \phi(B) .$$

Note also that $B(x_2) = B(x_1)$. Thus from (2) we find, with $U \frac{dB}{dx} = \eta$, $dH_2 = (-1 + UB)dB$, and free choice of a constant,

$$(4) \quad \frac{1}{2} \eta^2 + H_2(B) = 0 \quad \text{for } x \geq x_2 .$$

The crux of our argument is the following. If the solution behind the shock is to be eventually constant, it must tend for large x to satisfy $\eta = -1 + UB = 0$, (i.e. $E = -U \times B$ and $\frac{dB}{dx} = 0$). In other words it must pass through

$$\eta = 0 , \quad \frac{dH_2}{dB} = 0 .$$

In the neighborhood of this point in the B, η plane $\frac{1}{2} \eta^2 + H_2(B) = 0$ must be a pair of straight lines. (The case where the curves degenerate to a point corresponds to weak shocks, see [3] and cannot be attained without resistance.)

Thus near the end state with magnetic field $B = B^*$ we must have

$$(5) \quad H_2(B) < 0$$

and at the end state

$$H_2(B^*) = 0$$

(6)

$$-1 + U(B^*)B^* = 0 .$$

The function $H_2(B)$ is uniquely determined by (4) by giving the values of B , η at $x = x_2$. These in turn are given by B , $\frac{dB}{dx}$, and U at $x = x_2$. But B , $\frac{dB}{dx}$ are constant across $x_1 < x < x_2$ and hence are determined at x_1 by $\phi_{\max}$. $U|_{x=x_2}$ is determined by $R_2(\phi)$ hence by $\phi(x_2)$ and $\phi_{\max}$ but $\phi(x_2) = \phi_{\max}$ provided ϕ is decreasing for $x > x_2$. Hence B^* is determined by $\phi_{\max}$ from the equation $H_2(B^*) = 0$. But also from (6), $-1 + U(B^*)B^* = 0$. Hence there is one condition on $\phi_{\max}$ which determines it.

Hence the final value of B is determined by the requirement that we end in a constant state. (This idea appeared first in Morton [3].)

At the same time the final values of ϕ , ρ , U etc. are all determined.

We examine the requirement that ϕ is decreasing and the meaning of the condition (5), $H_2(B) < 0$, see (4), or by (1)

$$H_2 = \int_{B^*}^B (-1 + UB) dB = -(B - B^*) + \phi - \phi^* .$$

The question reduces to:

$$\text{when is } \left[B - \phi \right]_{B^*}^B > 0 ?$$

At $B = B^*$, $\frac{d\phi}{dB} = 1$ and so we compute $\frac{d^2\phi}{dB^2}$. If this quantity is negative then we have our desired type of endpoint.

From the momentum equation, (Part II, (I) for $\bar{u}$ independent of t), $\rho U^2 + (M_1 + \varepsilon^2 M_2)_x + \frac{B^2}{2} = \text{const.}$ we can determine the jump across the "black box" in electron momentum. We write the total pressure $M_1 + \varepsilon^2 M_2$ as p and regard it, as we may, as a function of x and hence of ϕ and hence of ρ . We denote

$$\frac{dp}{d\rho} = c^2$$

Then since $\rho U = 1$

$$dU + c^2 d\rho + BdB = 0$$

or

$$dU - c^2 \frac{dU}{U^2} + BdB = 0 .$$

Hence $\frac{dU}{dB} = \frac{-BU^2}{U^2 - c^2}$. But $UBdB = d\phi$ by the current equations. Hence

$$\frac{d^2\phi}{dB^2} = U + B \frac{dU}{dB} = U - \frac{B^2 U^2}{U^2 - c^2} = U(U^2 - c^2 - A^2)/(U^2 - c^2)$$

where $A = \sqrt{B^2/\rho}$ is the Alfvén speed behind the shock. Hence $\frac{d^2\phi}{dB^2}$ is positive if the flow speed is less than gas "sonic speed" as defined here. Lastly, $\frac{d\phi}{dB} < 0$ in $x_2 < x < \infty$ provided $(c^2 - U^2 - A^2)/(c^2 - U^2)$ is positive. This will hold if "the black box" represents a sufficiently strong internal shock. Equation (4) guarantees that $\frac{dB}{dx} = \frac{\eta}{U}$ does not change sign so that $\frac{d\phi}{dx}$ is negative for $x_2 < x < \infty$. Hence $\phi_{\max} = \phi(x_2)$ as desired. Thus the configuration agrees with the conditions used to describe the various zones. The chief property to note is the very weak assumptions about the motion. The worst feature is of course the neglect of

the gradient of the electron pressure in determining the current [cf. (1) and the equation for E, equation (I) in Part II]. Note that the exponential rate of approach to equilibrium behind the shock, e^{-Kx} , is given by $K = \left. \sqrt{\frac{d^2\phi}{dB^2}} \right|_{B=B^*} = U \sqrt{(A^2 + C^2 - U^2)/(C^2 - U^2)} \Big|_{B=B^*}$. Thus the width of the final region scales like the density ratio for very large Mach numbers since $U \sim \rho^{-1}$.

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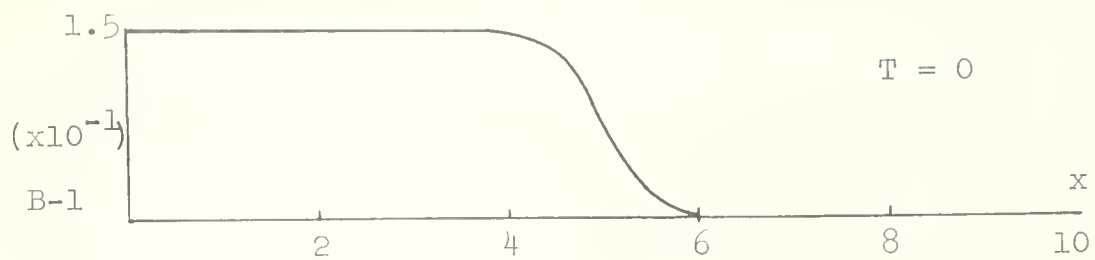


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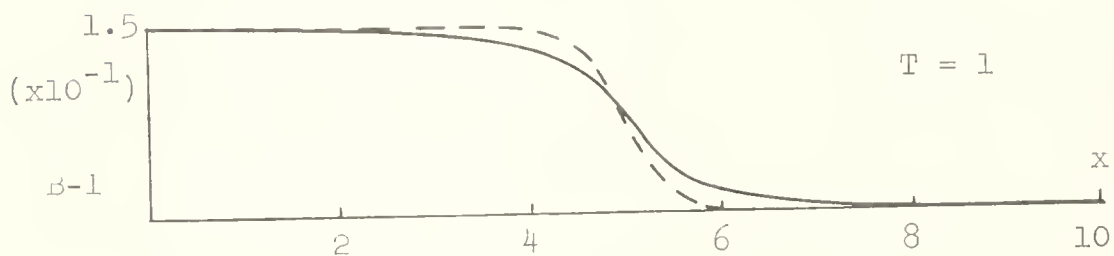


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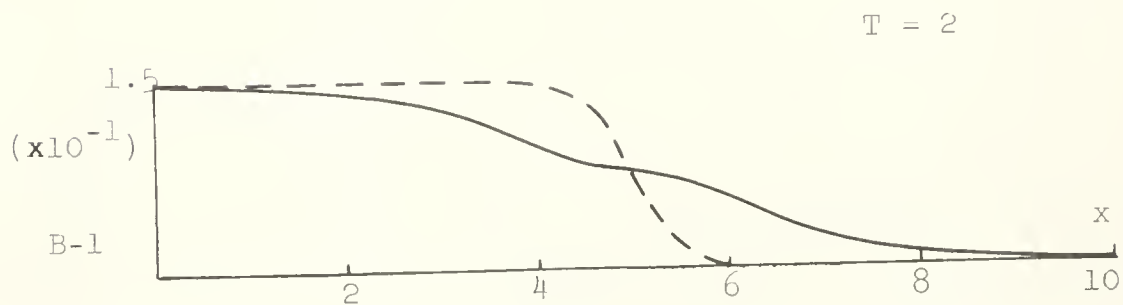


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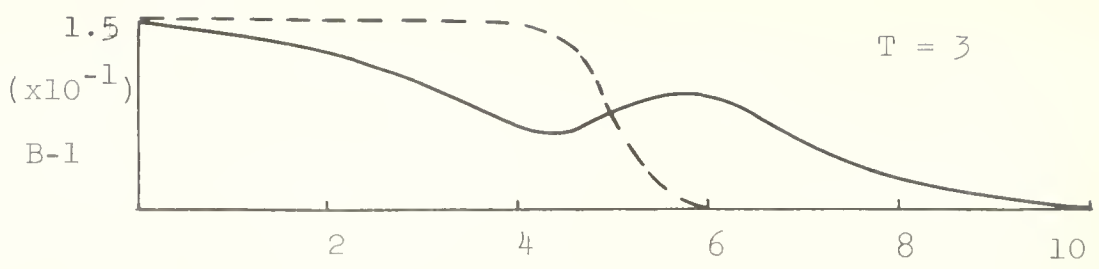


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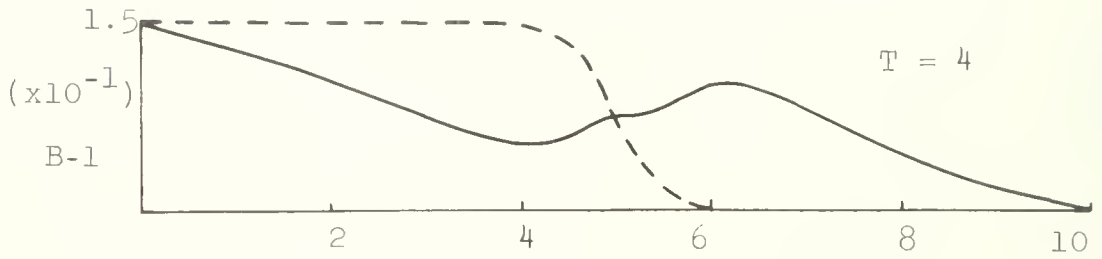


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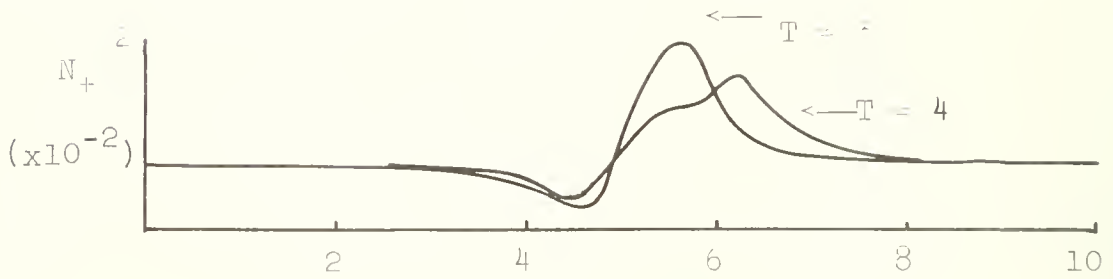


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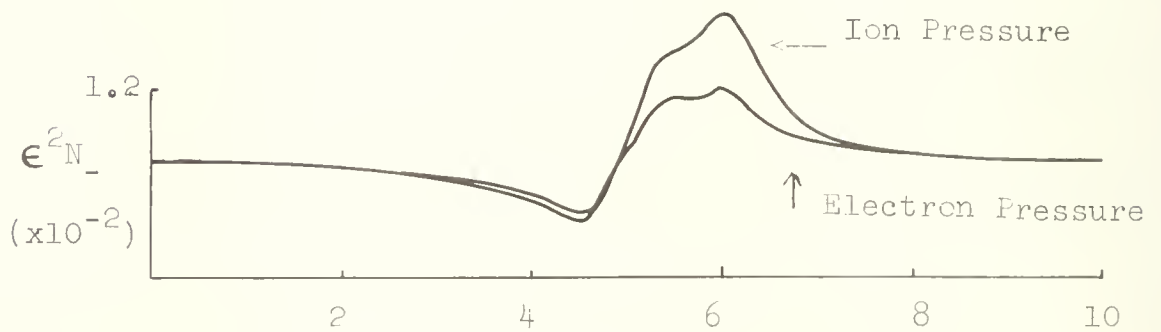
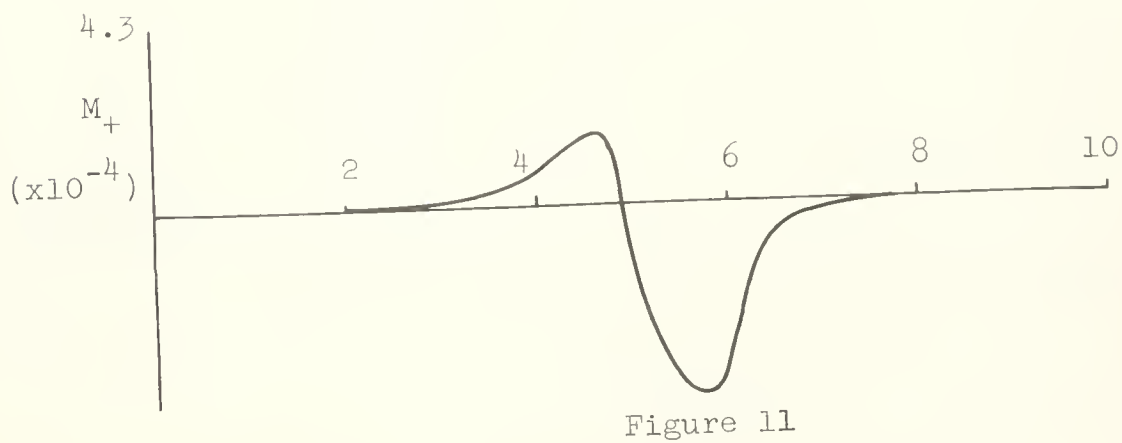
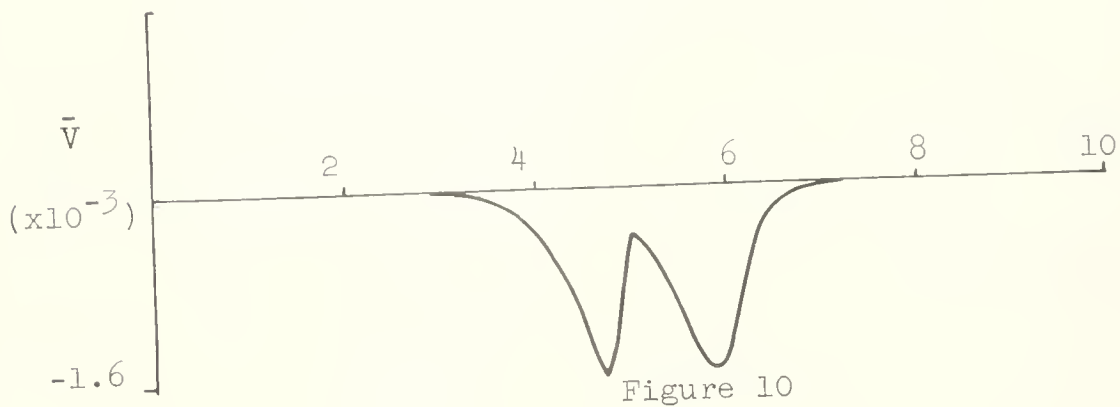
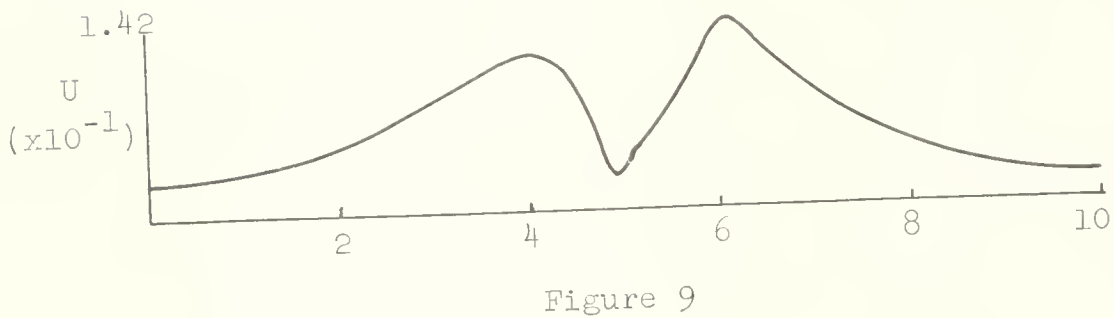
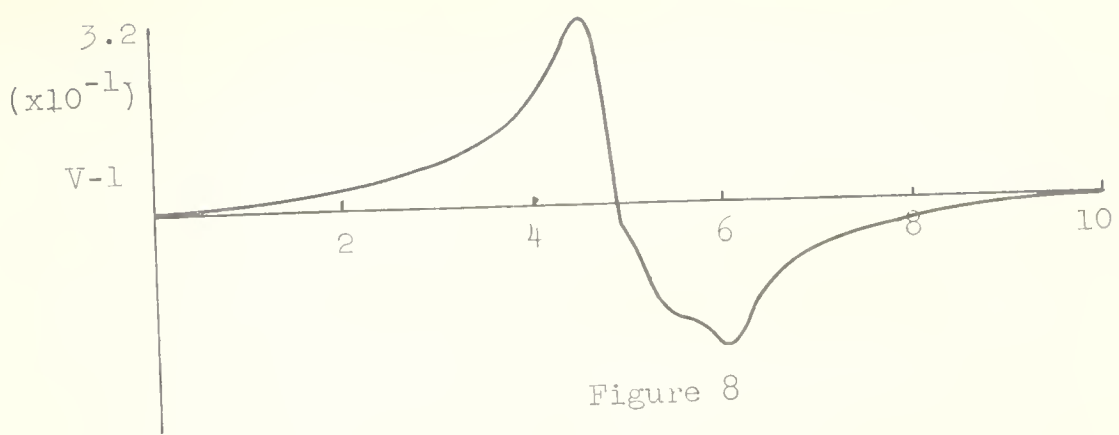


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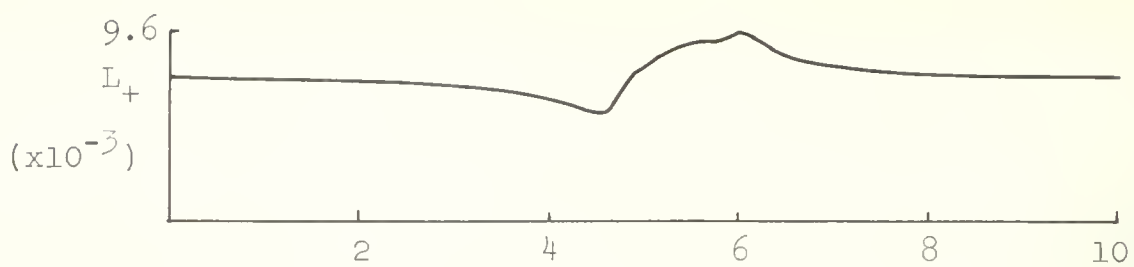


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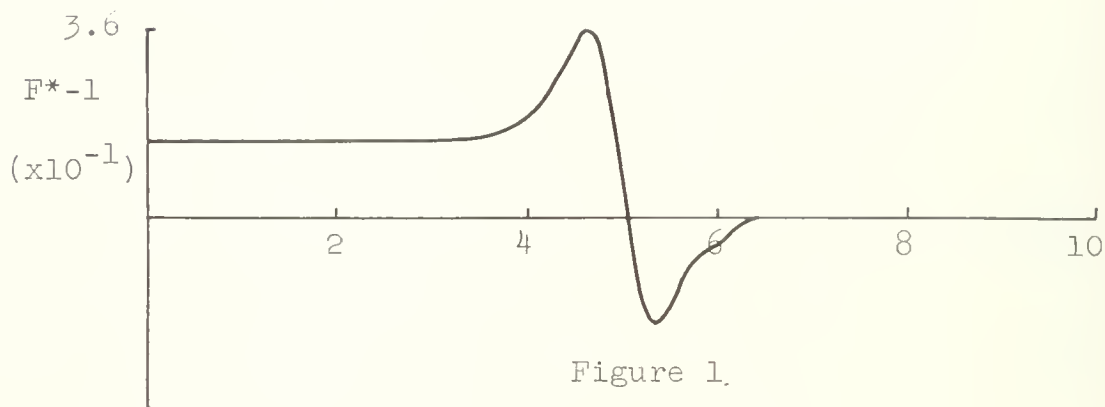


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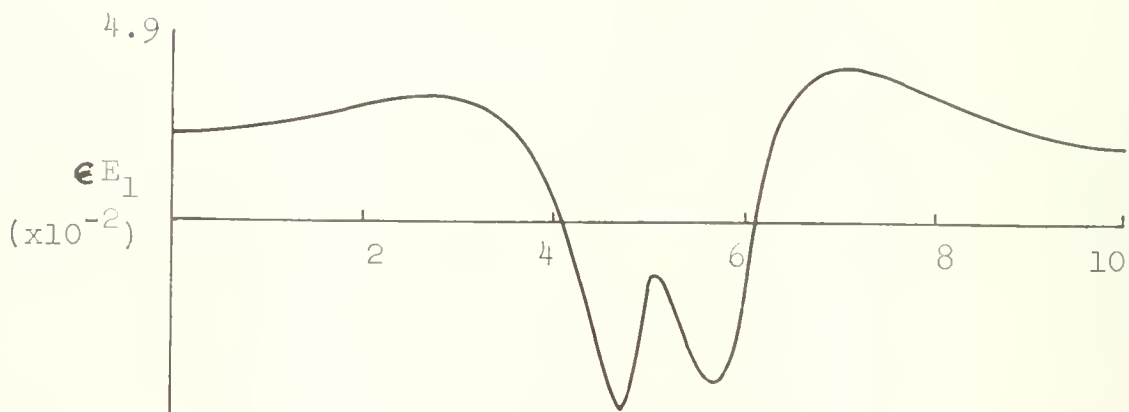


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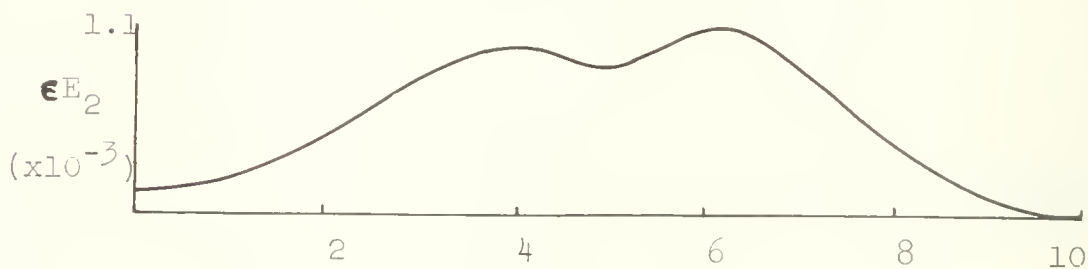
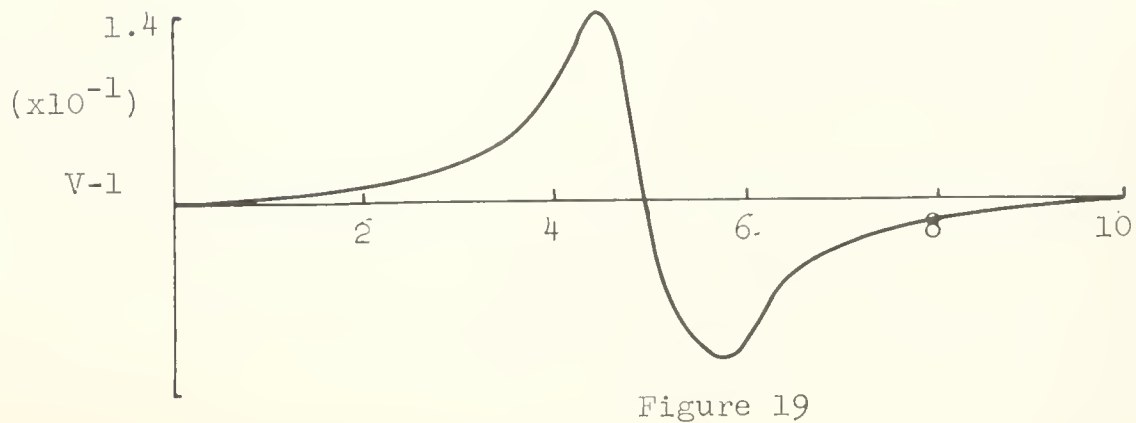
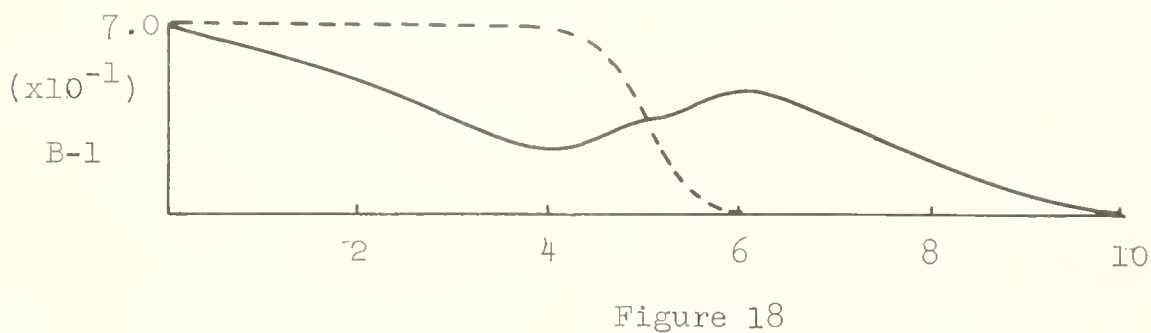
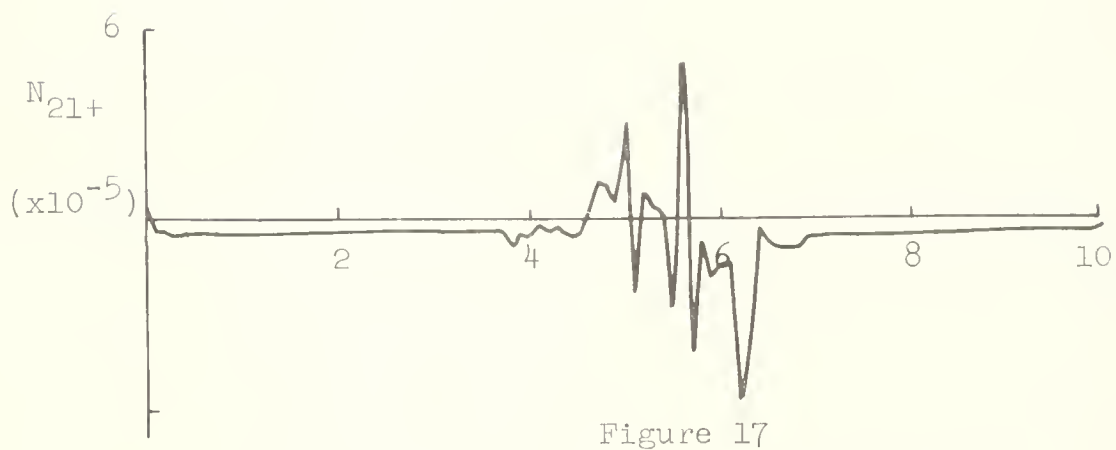
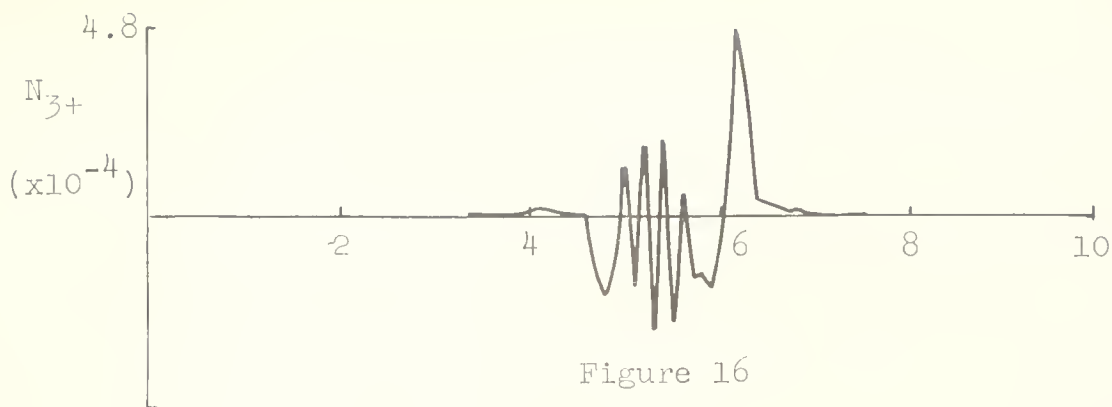


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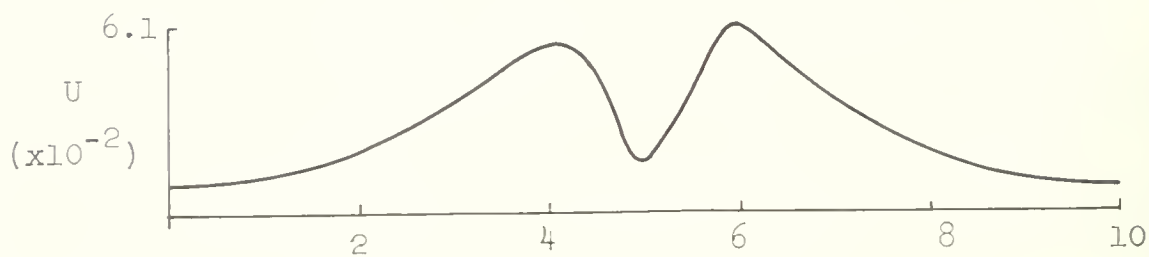


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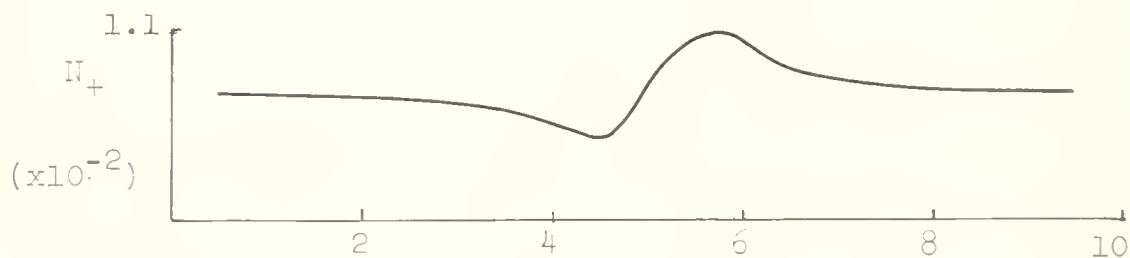


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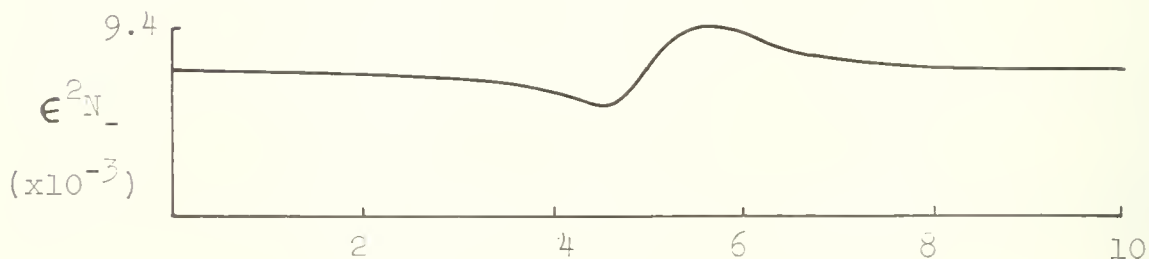


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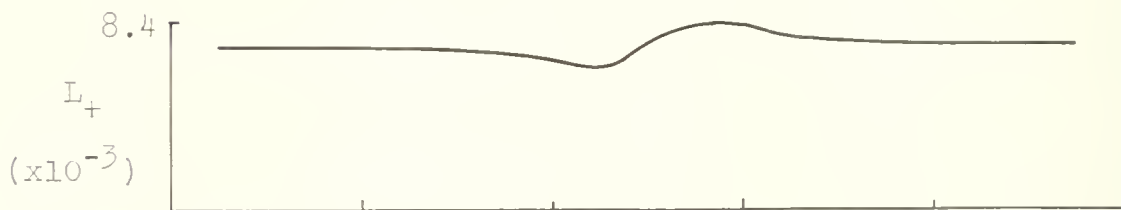


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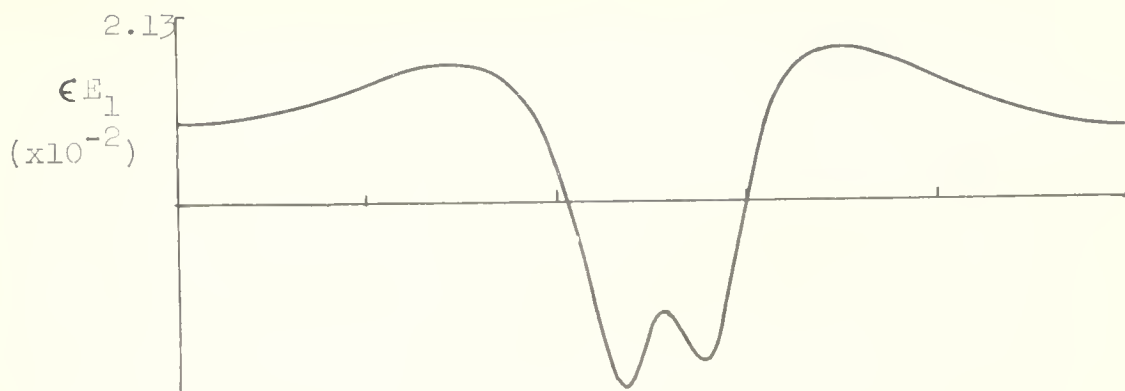


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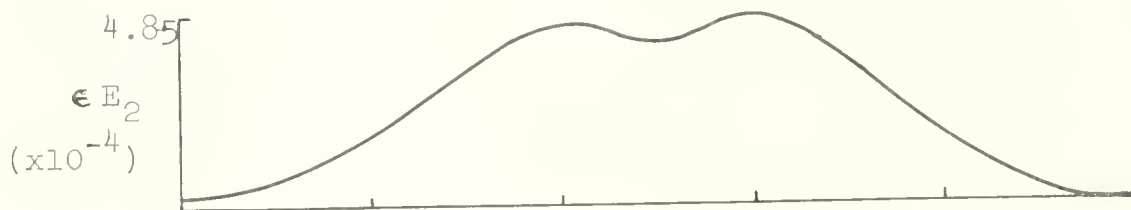


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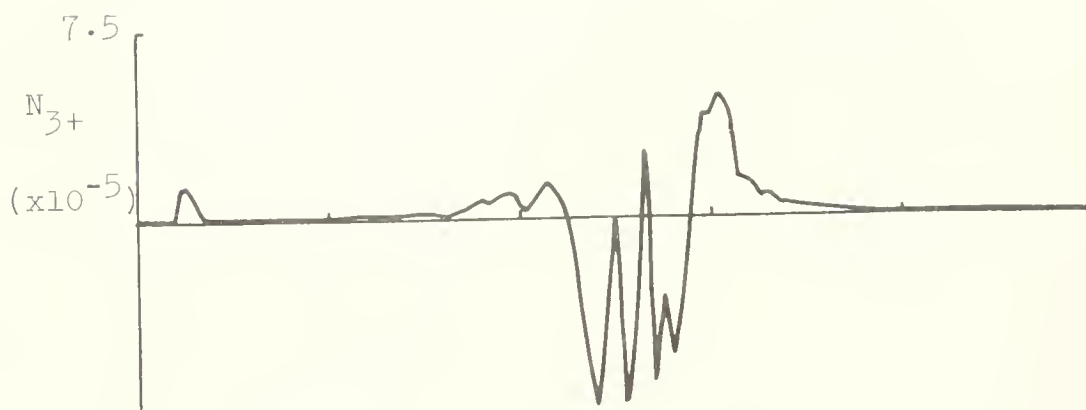


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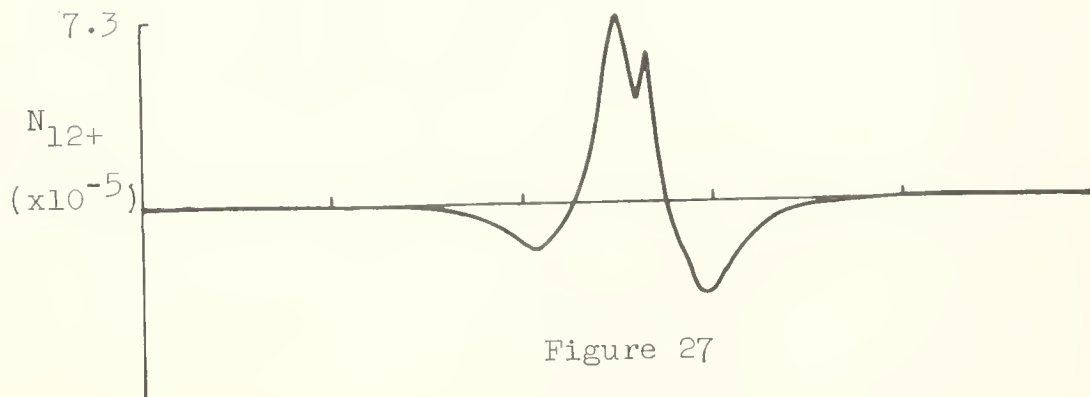
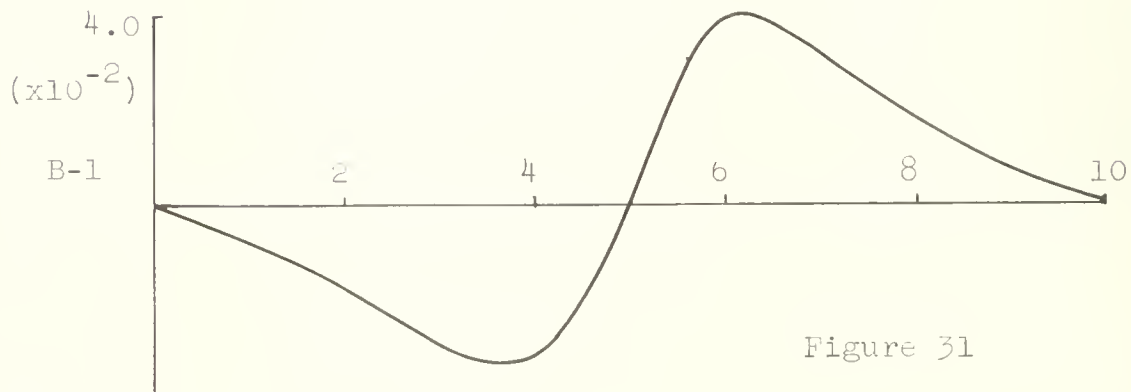
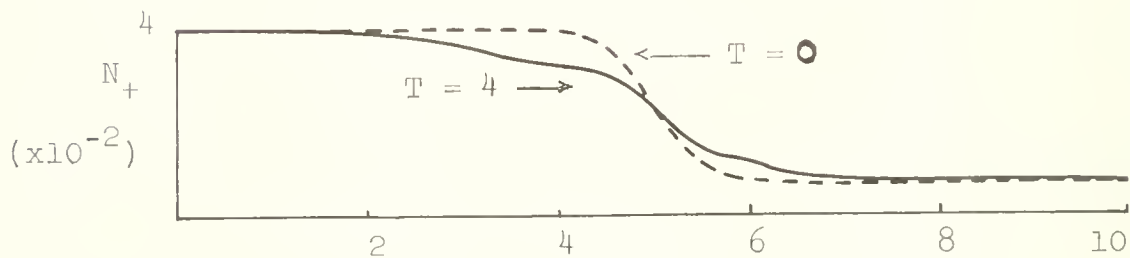
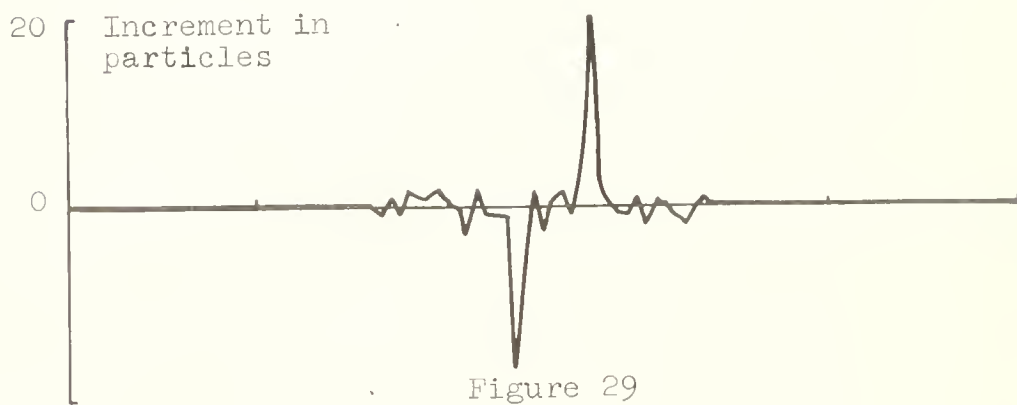
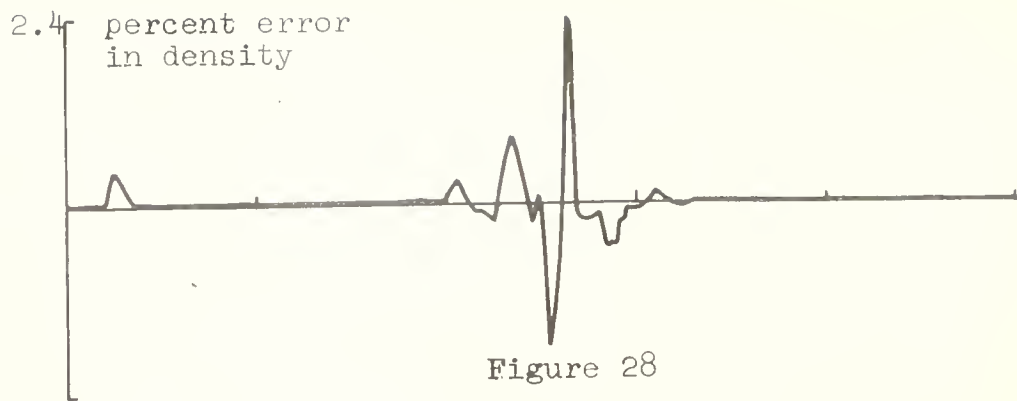
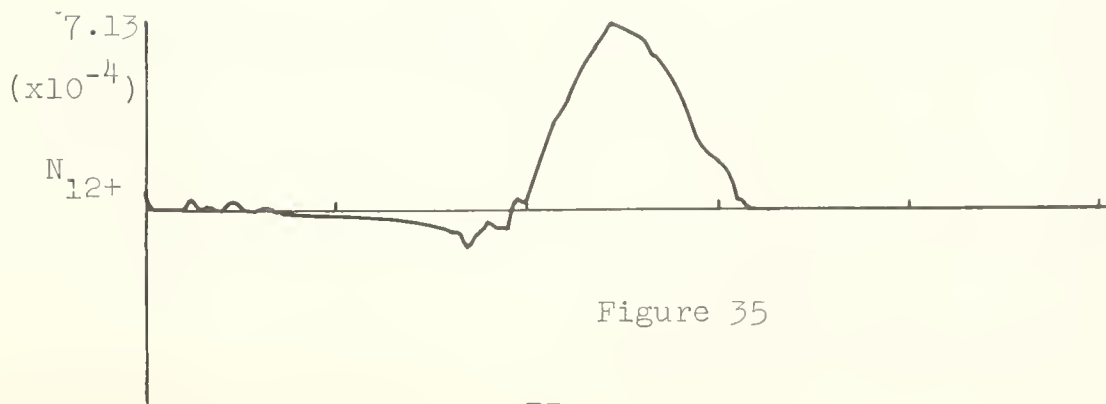
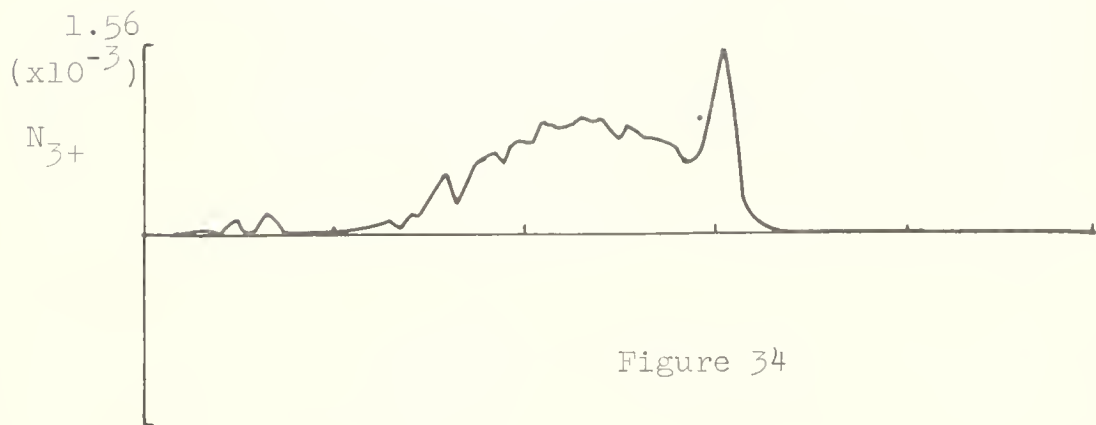
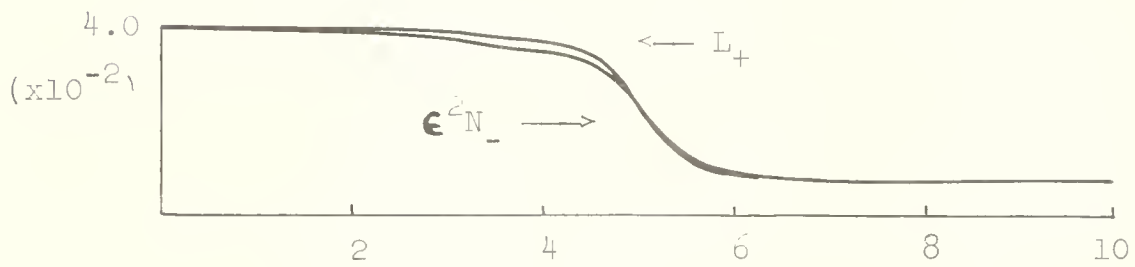
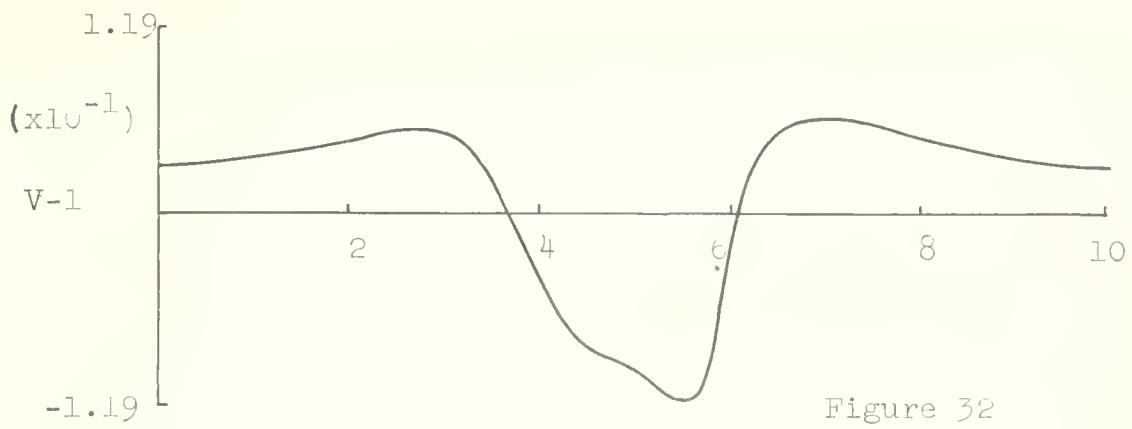


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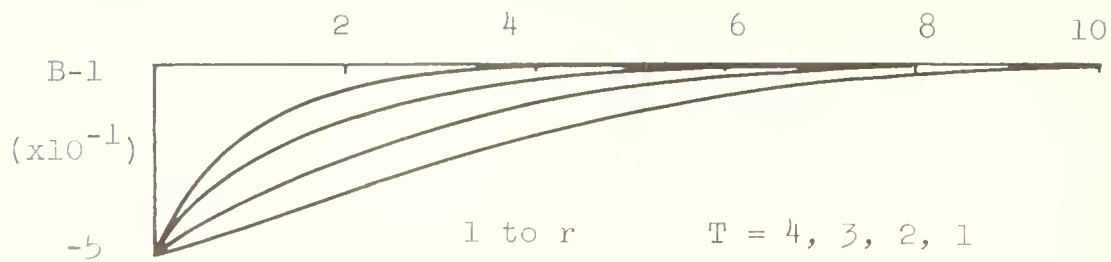


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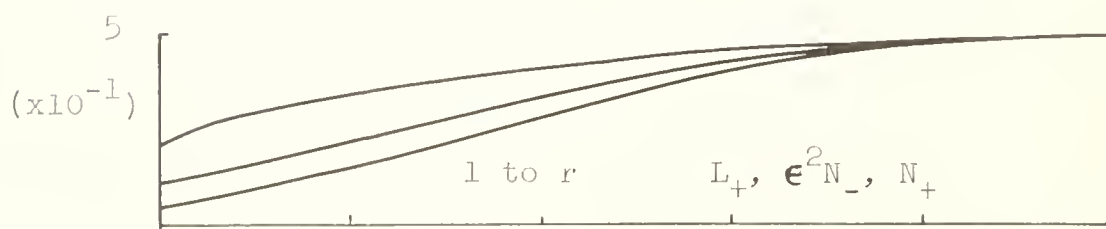


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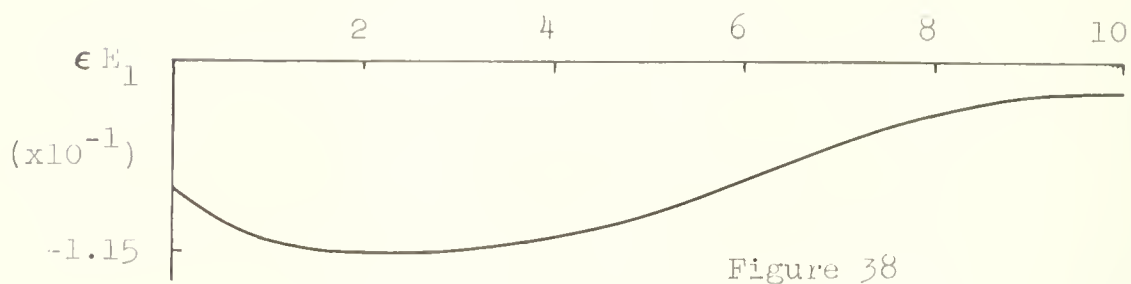


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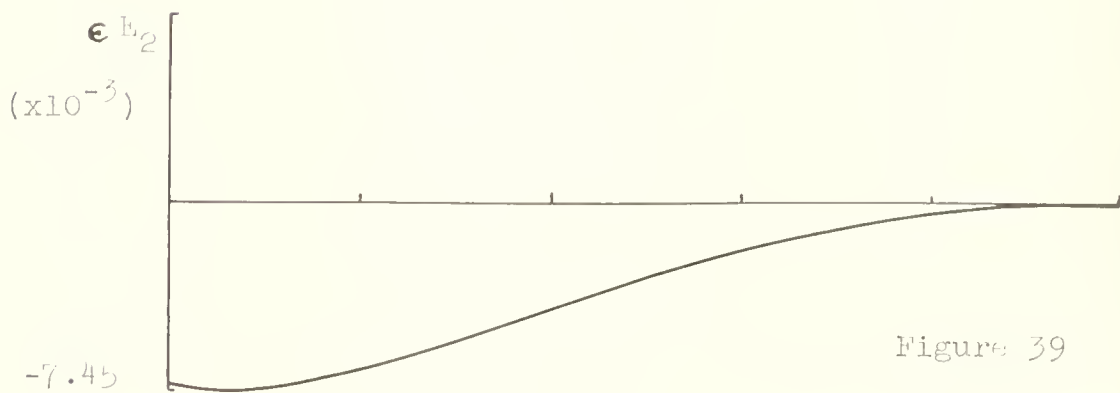


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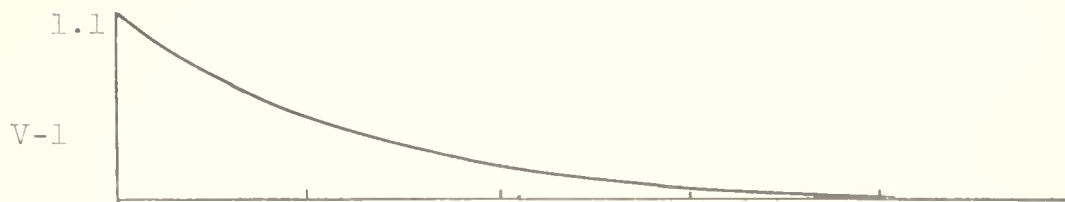


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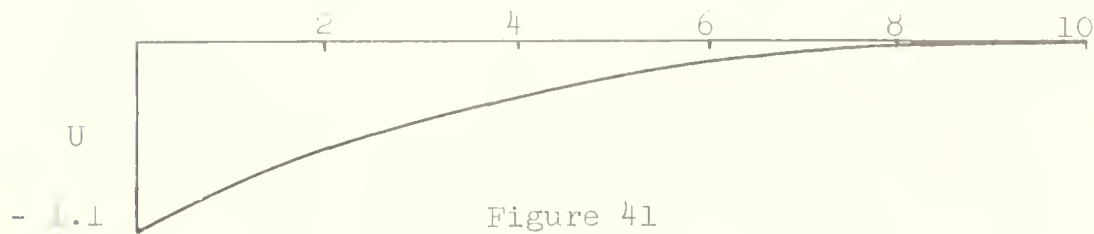


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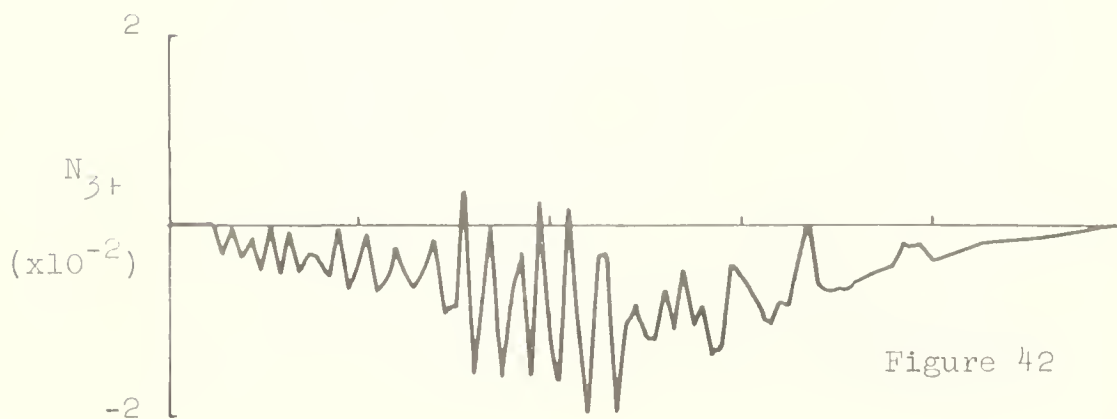


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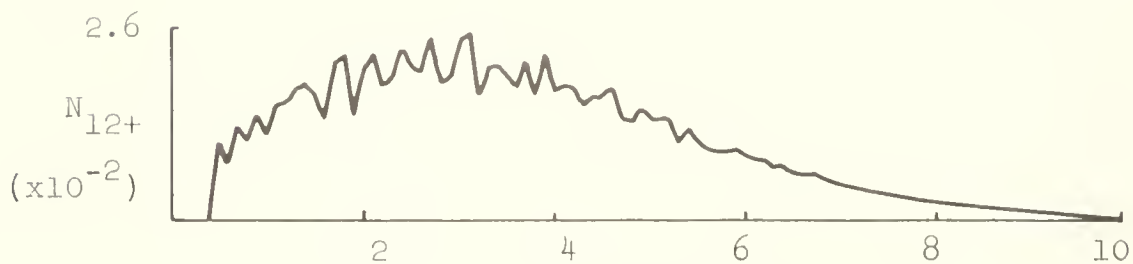
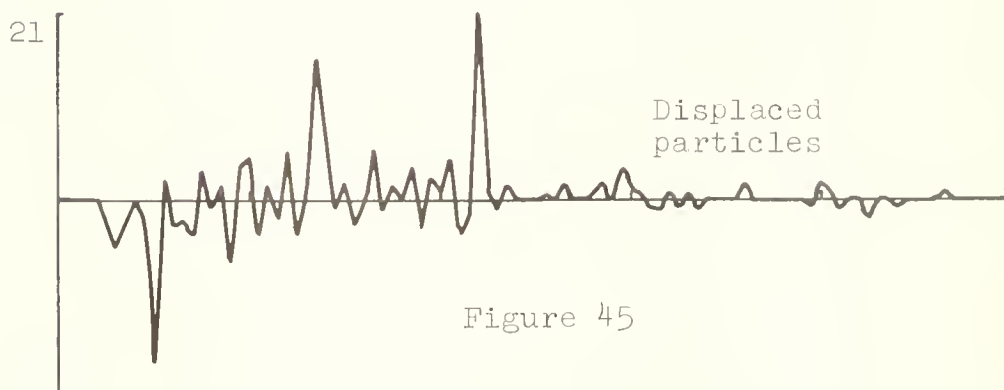
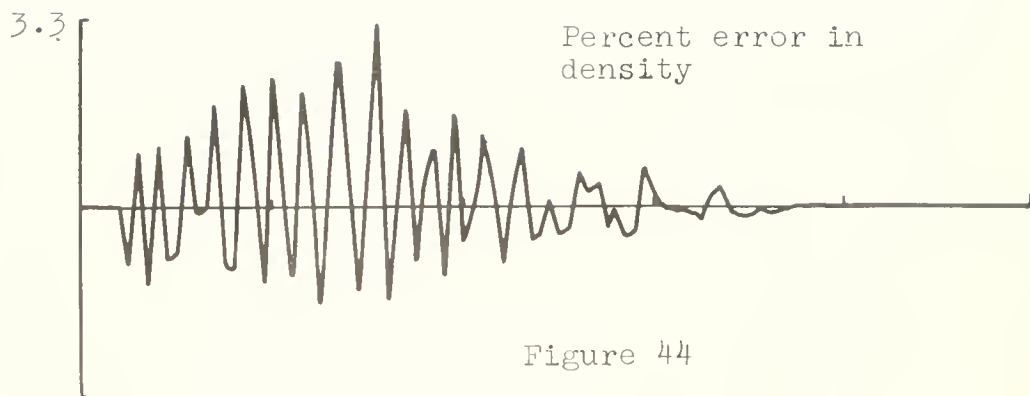


Figure 43



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